

A CONVEXITY-PRESERVING AND PERIMETER-DECREASING PARAMETRIC FINITE ELEMENT METHOD FOR THE AREA-PRESERVING CURVE SHORTENING FLOW*

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Abstract. We propose and analyze a semidiscrete parametric finite element scheme for solving the area-preserving curve shortening flow. The scheme is based on Dziuk’s approach [*SIAM J. Numer. Anal.*, 36 (1999), pp. 1808–1830] for the anisotropic curve shortening flow. We prove that the scheme preserves two fundamental geometric structures of the flow with an initially convex curve: (i) the convexity-preserving property, and (ii) the perimeter-decreasing property. To the best of our knowledge, the convexity-preserving property of numerical schemes which approximate the flow is rigorously proved for the first time. Furthermore, the error estimate of the semidiscrete scheme is established, and numerical results are provided to demonstrate the structure-preserving properties as well as the accuracy of the scheme.

Key words. area-preserving curve shortening flow, parametric finite element method, error estimate, convexity-preserving, perimeter-decreasing

MSC codes. 65M60, 65M12, 53C44, 35K55

DOI. 10.1137/22M1514404

1. Introduction. Consider the volume-preserving mean curvature flow driven by the motion law

$$(1.1) \quad v = (H - \langle H \rangle) \mathcal{N} \quad \text{on} \quad \Gamma_t,$$

where Γ_t is a family of smooth hypersurfaces in \mathbb{R}^n , v denotes the velocity, \mathcal{N} is the inner normal vector, H represents the scalar mean curvature (with the sign convention that H is positive for balls), and $\langle H \rangle := \int_{\Gamma_t} H ds^{n-1} / \int_{\Gamma_t} ds^{n-1}$ is the average mean curvature along Γ_t . It is well known that the volume-preserving mean curvature flow can be interpreted as the L^2 -gradient flow of the area functional under configurations with a fixed volume [37]. The volume-preserving mean curvature flow has the following fundamental geometric properties:

- (i) *Volume-preserving* [2, Lemma 5.25]. It can be immediately verified that the volume enclosed by Γ_t is indeed preserved by noticing

$$\frac{d}{dt} |\Omega_t| = - \int_{\Gamma_t} v \cdot \mathcal{N} ds^{n-1} = - \int_{\Gamma_t} (H - \langle H \rangle) ds^{n-1} = 0,$$

where Ω_t is the region enclosed by Γ_t . In dimension two (i.e., $n = 2$), it becomes the *area-preserving* property for a planar curve.

*Received by the editors August 8, 2022; accepted for publication (in revised form) March 24, 2023; published electronically August 23, 2023.

<https://doi.org/10.1137/22M1514404>

Funding: The work of the first author was supported by the National Natural Science Foundation of China (NSFC) grants 12271414 and 11871384. The work of the second author was supported by the NSFC grant 12201342.

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(ii) *Area-shrinking* [2, Lemma 5.25]. Actually, one can easily check that

$$\frac{d}{dt}|\Gamma_t| = - \int_{\Gamma_t} H v \cdot \mathcal{N} ds^{n-1} = - \int_{\Gamma_t} (H - \langle H \rangle)^2 ds^{n-1} \leq 0.$$

When $n = 2$, it becomes *perimeter-decreasing* for a planar curve.

(iii) *Convexity-preserving*. When $n = 2$, it was shown by Gage that starting from an initially smooth and convex closed curve, this flow (1.1) *preserves the convexity* and evolves the curve into a circle [23]. Furthermore, Huisken extended the result to higher dimensional cases [25]. For more general initial data, interested readers may refer to [1, 22].

In this paper, we focus on the planar curve ($n = 2$). In this case, the volume-preserving mean curvature flow is also known as the area-preserving curve shortening flow (AP-CSF), and it can be parametrized by the following equation [23]:

$$(1.2) \quad \begin{cases} \partial_t X = (H - \frac{2\pi}{L}) \mathcal{N}, & \xi \in \mathbb{S}^1, \quad t \in (0, T], \\ X(\xi, 0) = X^0(\xi), & \xi \in \mathbb{S}^1, \end{cases}$$

where $X(\xi, t) : \mathbb{S}^1 \times [0, T] \rightarrow \Gamma_t \subseteq \mathbb{R}^2$, $L := L(t)$ is the length of Γ_t , by recalling the theorem of turning tangents [17], i.e., $\int_{\Gamma_t} H ds^1 = 2\pi$, for a simple closed curve Γ_t .

Nowadays, the AP-CSF has found important applications in many research areas, such as materials science and image processing [27], and it can be viewed as an area-preserving variant of the CSF [35, 42] or a limit flow of the nonlocal Ginzburg–Landau equation [13]. There have been extensive numerical investigations concerning the CSF or AP-CSF in the last decades. Among them, parametric finite element methods (PFEMs) have been widely proposed for simulating the CSF and some other related geometric flows [3], e.g., the surface diffusion flow [4], and anisotropic geometric flows [6, 8, 9, 10]. Numerical approximations to the CSF by using PFEMs could date back to the pioneering work of Dziuk [18] in 1991. Since then, various techniques have been introduced to make the designed PFEMs more accurate and efficient in practical simulations, including the method of Barrett, Garcke, and Nürnberg (the BGN scheme) [6, 7] based on a novel variational formulation, the method of Deckelnick and Dziuk by introducing an artificial tangential velocity [15], and the method proposed by Elliott and Fritz based on special reparametrizations [21]. These methods induce appropriate tangential motions that lead to good mesh distribution properties, which play a vital role in numerical simulations. Recently, more and more attention has been paid to designing “structure-preserving” (e.g., area-preserving or perimeter-decreasing) PFEMs for solving geometric evolution flows [3, 4, 28].

However, error estimates for these schemes seem difficult and quite challenging. For example, Dziuk first studied the convergence of a semidiscrete linearly implicit PFEM for the CSF [19] and anisotropic CSF [20], respectively, based on a finite difference structure; Li developed a new technique to analyze the convergence of semidiscrete high-order PFEMs for the CSF [33] and mean curvature flow of closed surfaces [34], respectively. For Dziuk’s fully discrete linearly implicit scheme [19], very recently an optimal error estimate in H^1 was established by Ye and Cui [44]. As for the error analysis about other numerical methods of the CSF or other related geometric flows, we refer the reader to [5, 15, 21, 26, 30, 31, 39].

As for the AP-CSF, there exist various numerical methods in the literature, e.g., the finite difference method [35], the MBO method [29, 40], the crystalline algorithm [43], and PFEMs [11, 38]. Particularly, structure-preserving properties were investigated in [11, 38, 41, 43]. For example, the semidiscrete PFEM in [11] based

on an elegant variational formulation was shown to preserve the length shortening property. The semidiscrete polygonal evolution law in [41] based on the definition of tangent and normal vectors/velocities at each vertex and the fully discrete PFEM in [38] were shown to be area-preserving and perimeter-decreasing. However, error estimates have been barely studied for the above-mentioned methods, except for the crystalline algorithm in [43], where the error estimate was only established for the curvature since the numerical scheme was designed based on the crystalline approximation, which merely involves the curvature. To the best of our knowledge, there exist few numerical analyses of numerical methods for solving the AP-CSF, and the convexity-preserving property has never been investigated in the literature. The reason is that the nonlocal term in the AP-CSF has brought about difficulties and considerable challenges in numerical analysis.

In this paper, we propose a semidiscrete PFEM for the AP-CSF based on Dziuk's approach for anisotropic CSF [20], investigate its structure-preserving properties, and present its error analysis. Specifically, we prove that our scheme preserves two important geometric structures of the AP-CSF, i.e., convexity-preserving and perimeter-decreasing properties. As far as we know, this is the first work to rigorously prove the convexity-preserving property and to give the error estimate of numerical methods for solving the AP-CSF.

To start, (1.2) can be written more explicitly as

$$(1.3) \quad \partial_t X = \frac{1}{|\partial_\xi X|} \partial_\xi \left(\frac{\partial_\xi X}{|\partial_\xi X|} \right) - \frac{2\pi}{L} \left(\frac{\partial_\xi X}{|\partial_\xi X|} \right)^\perp,$$

where $(a, b)^\perp := (-b, a)$. This naturally yields a weak formulation: for any $v \in (H^1(\mathbb{S}^1))^2$, it holds that

$$(1.4) \quad \int_{\mathbb{S}^1} |\partial_\xi X| \partial_t X \cdot v \, d\xi + \int_{\mathbb{S}^1} \frac{\partial_\xi X}{|\partial_\xi X|} \cdot \partial_\xi v \, d\xi + \int_{\mathbb{S}^1} \frac{2\pi}{L} (\partial_\xi X)^\perp \cdot v \, d\xi = 0.$$

As mentioned in [19], the derived linearly implicit PFEM from the above formulation for the CSF (with the last term missing) may fail to preserve the length shortening property of the CSF. To overcome this, Dziuk proposed another scheme based on the lumping of masses in [19] for the CSF. Here we utilize a similar approach: find a solution $X_h(\xi, t) \in V_h \times [0, T]$ satisfying the weak form (2.4) with initial condition $X_h(\xi, 0) = I_h X^0$, where V_h is a vector-valued Lagrange finite element space consisting of a piecewise linear polynomial and I_h is the standard Lagrange interpolation. Similar to [14], the semidiscrete scheme focuses on the motion of the initial polygon, which is determined by the evolution of the vertices. We show that if the initial curve is convex, then the evolved polygon remains convex all the time. Moreover, the perimeter of the polygon is decreasing. To show the convexity-preserving property, we characterize the convexity of a polygon by the positivity of the oriented area of all adjacent triangles, which will be shown by a contradiction argument. Surprisingly, the perimeter-decreasing property can be reduced to a pure trigonometric inequality when the polygon remains convex. We note that nondegeneration of vertices is necessary to ensure the evolved polygons are well behaved. This will be guaranteed by the error estimate of the scheme, which shows that the semidiscrete scheme (2.4) converges in H^1 at the first order, and the lower bound of the edge lengths of the polygon could stay positive all the time.

The rest of the paper is organized as follows. In section 2, we start with the spatial discretization which approximates the AP-CSF and summarize our main results.

In section 3, we prove that the numerical scheme rigorously preserves two important geometric structures of the flow, i.e., the convexity-preserving and perimeter-decreasing properties. Then, we present the proof of the error estimate of the scheme in section 4. Finally, some numerical results produced by the scheme are provided in section 5 to validate our theoretical results.

2. Spatial discretization and main results. Let $0 = \xi_0 < \xi_1 < \dots < \xi_N = 2\pi$ be a partition of $\mathbb{S}^1 = [0, 2\pi]$. We denote $h_j = \xi_j - \xi_{j-1}$ by the length of the interval $I_j := [\xi_{j-1}, \xi_j]$ and $h = \max_j h_j$. Throughout the paper, we use a periodic index, i.e., $f_j = f_{j \pm N}$ when involved. We assume that the partition and the exact solution are regular in the following senses, respectively.

Assumption 2.1. There exist constants C_p and C_P such that

$$\min_j h_j \geq C_p h, \quad |h_{j+1} - h_j| \leq C_P h^2, \quad 1 \leq j \leq N.$$

Assumption 2.2. Suppose that the unique solution of (1.2) with an initial value $X^0 \in H^2(\mathbb{S}^1)$ satisfies $X \in W^{1,\infty}([0, T], H^2(\mathbb{S}^1))$, i.e.,

$$K(X) := \|X\|_{W^{1,\infty}([0, T], H^2(\mathbb{S}^1))} < \infty.$$

We further assume that there exist constants $0 < \kappa_1 < \kappa_2$ such that

$$\kappa_1 \leq |\partial_\xi X(\xi, t)| \leq \kappa_2 \quad \forall (\xi, t) \in \mathbb{S}^1 \times [0, T].$$

We define the following finite element space consisting of piecewise linear functions satisfying periodic boundary conditions:

$$V_h = \{v \in C^0(\mathbb{S}^1, \mathbb{R}^2) : v|_{I_j} \in P_1(I_j), \quad 1 \leq j \leq N, \quad v(\xi_0) = v(\xi_N)\},$$

where P_1 denotes all polynomials with degrees at most 1. For any continuous function $v \in C^0(\mathbb{S}^1, \mathbb{R}^2)$, the linear interpolation $I_h v \in V_h$ is uniquely determined through $I_h v(\xi_j) = v(\xi_j)$ for all $1 \leq j \leq N$ and can be explicitly written as $I_h v(\xi) = \sum_{j=1}^N v(\xi_j) \varphi_j(\xi)$, where φ_j represents the standard Lagrange basis function satisfying $\varphi_j(\xi_i) = \delta_{ij}$. We have the following basic estimates from finite element theory.

LEMMA 2.1 (see [12]). *Under Assumption 2.1, there exists a constant C depending on C_p, C_P such that the following estimates hold:*

(i) *Interpolation estimate.* For any $Y \in H^2(\mathbb{S}^1)$, we have

$$(2.1) \quad \begin{aligned} \|Y - I_h Y\|_{L^2} &\leq Ch^k \|Y\|_{H^k}, \quad k = 1, 2; \quad \|Y - I_h Y\|_{L^\infty} \leq Ch^{1/2} \|Y\|_{H^1}, \\ \|\partial_\xi(Y - I_h Y)\|_{L^2} &\leq Ch \|Y\|_{H^2}, \quad \|\partial_\xi I_h Y\|_{L^2} \leq C \|Y\|_{H^1}. \end{aligned}$$

(ii) *Inverse estimate.* For $v_h \in V_h$, we have

$$(2.2) \quad \|v_h\|_{L^\infty} \leq Ch^{-1/2} \|v_h\|_{L^2}, \quad \|v_h\|_{H^1} \leq Ch^{-1} \|v_h\|_{L^2}.$$

DEFINITION 2.2. *We call a function*

$$(2.3) \quad X_h(\xi, t) = \sum_{j=1}^N X_j(t) \varphi_j(\xi) : \mathbb{S}^1 \times [0, T] \rightarrow \mathbb{R}^2$$

a semidiscrete solution of (1.3) if it satisfies the following weak formulation:

$$(2.4) \quad \int_{\mathbb{S}^1} |\partial_\xi X_h| \partial_t X_h \cdot v_h \, d\xi + \int_{\mathbb{S}^1} \frac{\partial_\xi X_h}{|\partial_\xi X_h|} \cdot \partial_\xi v_h \, d\xi + \int_{\mathbb{S}^1} \frac{\mathbf{h}^2 |\partial_\xi X_h|}{6} \partial_\xi \partial_t X_h \cdot \partial_\xi v_h \, d\xi + \int_{\mathbb{S}^1} \frac{2\pi}{L_h} (\partial_\xi X_h)^\perp \cdot v_h \, d\xi = 0 \quad \forall v_h \in V_h,$$

with initial condition $X_h(\xi, 0) = I_h X^0$, where L_h represents the perimeter of the evolved curve (image of X_h), i.e.,

$$L_h := \sum_{j=1}^N h_j |\partial_\xi X_h|_{I_j} = \sum_{j=1}^N |X_j - X_{j-1}| =: \sum_{j=1}^N q_j,$$

and \mathbf{h} is a piecewise constant $\mathbf{h} = h_j$ on I_j .

Remark 2.1. A similar version of (2.4) was proposed and analyzed in [19, (9) and (15)] and [20, Definition 4.1] for CSF and anisotropic CSF, respectively. The introduction of the third term $\int_{\mathbb{S}^1} \frac{\mathbf{h}^2 |\partial_\xi X_h|}{6} \partial_\xi \partial_t X_h \cdot \partial_\xi v_h \, d\xi$ in (2.4) gives rise to the so-called mass-lumped scheme (similar to (3.1)) that preserves the length shortening property for the CSF, which was missing for the original formulation (e.g., (1.4)). On the other hand, a more natural explanation was given in [39, (1.6) and (3.12)], where it was shown that (2.4) is equivalent to the following scheme:

$$\int_{\mathbb{S}^1} |\partial_\xi X_h| I_h (\partial_t X_h \cdot v_h) \, d\xi + \int_{\mathbb{S}^1} \frac{\partial_\xi X_h}{|\partial_\xi X_h|} \cdot \partial_\xi v_h \, d\xi + \int_{\mathbb{S}^1} \frac{2\pi}{L_h} (\partial_\xi X_h)^\perp \cdot v_h \, d\xi = 0 \quad \forall v_h \in V_h,$$

which looks like the original version (1.4) with the Lagrangian interpolation introduced for the first term.

Next we present the main results of this paper.

THEOREM 2.3 (convexity-preserving). *Suppose the initial curve $X_h(\xi, 0) = I_h X^0$ is a convex N -polygon; then it is always a convex N -polygon during the evolution by (2.4) if $q_j > 0$ for all j .*

THEOREM 2.4 (perimeter-decreasing). *Let X_h be the solution of (2.4) with convex initial data; then the perimeter of the closed curve is decreasing, i.e.,*

$$(2.5) \quad \frac{d}{dt} L_h \leq 0.$$

Remark 2.2. For the cases of classical CSF or anisotropic CSF and the corresponding solutions based on similar formulations as in (2.4), by direct computations based on a finite difference structure, it was shown in [19, 20] that the length of each element of X_h is decreasing, i.e., $q'_j(t) \leq 0$ for $1 \leq j \leq N$. This directly implies the perimeter-decreasing property. We point out that this property can also be obtained by standard energy estimates for the CSF. However, in our AP-CSF case, both arguments fail to derive the perimeter-decreasing property. We have to carry out a more careful investigation in which the convexity property plays a vital role (section 3.2).

Remark 2.3. We would like to point out that our scheme (2.4) cannot strictly mimic the area-preserving property of AP-CSF in the discrete form. As shown in Figure 5.1(b), our scheme has an area loss at $\mathcal{O}(h^2)$. The area-preserving property of AP-CSF can be achieved at the discrete level by using the BGN-type schemes as proposed in [4, 28, 38], but the question of how to prove the convexity-preserving property and establish the error estimates of the BGN-type schemes is still open.

THEOREM 2.5 (error estimate). *Let $X(\xi, t)$ be a solution of (1.3) satisfying Assumption 2.2. Assume that the partition of \mathbb{S}^1 satisfies Assumption 2.1. Then there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, there exists a unique semidiscrete solution X_h for (2.4). Furthermore, the solution satisfies*

$$\int_0^T \|\partial_t X - \partial_t X_h\|_{L^2}^2 ds + \sup_{[0, T]} \|X - X_h\|_{H^1}^2 \leq Ch^2,$$

where h_0 and C depend on $C_p, C_P, \kappa_1, \kappa_2, T$, and $K(X)$. In particular, we have

$$(2.6) \quad \min_j q_j(t) > 0 \quad \forall t \in [0, T].$$

3. Convexity-preserving and perimeter-decreasing properties. Similar to what was done in [16], we rewrite (2.4) into a lumped mass formulation. More precisely, taking

$$v_h = (\varphi_j, 0) = \left(\frac{\xi - \xi_{j-1}}{h_j} \chi_{I_j} + \frac{\xi_{j+1} - \xi}{h_{j+1}} \chi_{I_{j+1}}, 0 \right)$$

in (2.4), where χ is the characteristic function, calculations in [20] give

$$\begin{aligned} & \int_{\mathbb{S}^1} |\partial_\xi X_h| \partial_t X_h \cdot v_h \, d\xi + \int_{\mathbb{S}^1} \frac{\partial_\xi X_h}{|\partial_\xi X_h|} \cdot \partial_\xi v_h \, d\xi + \int_{\mathbb{S}^1} \frac{\mathbf{h}^2 |\partial_\xi X_h|}{6} \partial_\xi \partial_t X_h \cdot \partial_\xi v_h \, d\xi \\ &= \frac{q_j + q_{j+1}}{2} \dot{X}_j^{[1]} - (\mathcal{T}_{j+1}^{[1]} - \mathcal{T}_j^{[1]}), \end{aligned}$$

where $a^{[1]}$ denotes the first component of the vector $a \in \mathbb{R}^2$, and

$$\mathcal{T}_j := \frac{X_j - X_{j-1}}{|X_j - X_{j-1}|}, \quad \mathcal{N}_j = \left(\frac{X_j - X_{j-1}}{|X_j - X_{j-1}|} \right)^\perp.$$

For the last term involving the perimeter, we similarly compute

$$\begin{aligned} & \int_{\mathbb{S}^1} \frac{2\pi}{L_h} (\partial_\xi X_h)^\perp \cdot v_h \, d\xi \\ &= \int_{\xi_{j-1}}^{\xi_j} \frac{2\pi}{L_h} \frac{q_j}{h_j} \mathcal{N}_j \cdot \left(\frac{\xi - \xi_{j-1}}{h_j}, 0 \right) \, d\xi + \int_{\xi_j}^{\xi_{j+1}} \frac{2\pi}{L_h} \frac{q_{j+1}}{h_{j+1}} \mathcal{N}_{j+1} \cdot \left(\frac{\xi_{j+1} - \xi}{h_{j+1}}, 0 \right) \, d\xi \\ &= \frac{\pi}{L_h} q_j \mathcal{N}_j^{[1]} + \frac{\pi}{L_h} q_{j+1} \mathcal{N}_{j+1}^{[1]}. \end{aligned}$$

Similarly taking $v_h = (0, \varphi_j)$ yields the equation for the second component. Thus the weak formulation (2.4) is equivalent to the following lumped mass formulation:

$$(3.1) \quad \frac{q_j + q_{j+1}}{2} \dot{X}_j = \mathcal{T}_{j+1} - \mathcal{T}_j - \frac{\pi}{L_h} (q_j \mathcal{N}_j + q_{j+1} \mathcal{N}_{j+1}) = \mathcal{T}_{j+1} - \mathcal{T}_j - \frac{\pi}{L_h} (X_{j+1} - X_{j-1})^\perp.$$

Hence it remains to solve the ODE system (3.1), and the image of X_h is a polygon with $X_j(t)$ as the vertices.

For further studies, we derive some important formulae which will be used frequently. Straightforward calculations as in [16, Proposition 4.1], [20, Lemmas 3.1 and 4.2] and [39, Lemmas 2.4 and 3.2] lead to

$$(3.2) \quad \partial_t |\partial_\xi X| = -|\partial_\xi X| |\partial_t X|^2 + \partial_t X \cdot R |\partial_\xi X|,$$

$$(3.3) \quad \frac{d}{dt} q_j = -\frac{1}{q_j + q_{j+1}} |\mathcal{T}_{j+1} - \mathcal{T}_j|^2 - \frac{1}{q_j + q_{j-1}} |\mathcal{T}_{j-1} - \mathcal{T}_j|^2 + \mathcal{T}_j \cdot (R_j - R_{j-1})$$

$$(3.4) \quad = -\frac{q_j + q_{j+1}}{4} |\dot{X}_j - R_j|^2 - \frac{q_j + q_{j-1}}{4} |\dot{X}_{j-1} - R_{j-1}|^2 + \mathcal{T}_j \cdot (R_j - R_{j-1}),$$

where for simplicity we denote

$$(3.5) \quad R := -\frac{2\pi}{L} \mathcal{N}, \quad R_j := -\frac{2\pi}{L_h} \frac{\mathcal{N}_j q_j + \mathcal{N}_{j+1} q_{j+1}}{q_j + q_{j+1}}.$$

By using above quantities, (3.1) can also be written as

$$(3.6) \quad \dot{X}_j - R_j = 2(\mathcal{T}_{j+1} - \mathcal{T}_j) / (q_j + q_{j+1}).$$

In this section, we will prove that this semidiscrete geometric flow preserves the convexity of polygons under the nondegeneration property of vertices, which can be guaranteed by (2.6). Furthermore, the perimeter-decreasing property is also shown for convex initial data.

3.1. Proof of Theorem 2.3. First, we carry out some clarifications concerning a polygon. We denote $\mathcal{P} = (Y_1, \dots, Y_N)$ as an N -polygon with Y_j being its vertices and $\overline{Y_{j-1}Y_j}$ being the edge connecting Y_{j-1} and Y_j . We emphasize that $\mathcal{P} = (Y_1, \dots, Y_N)$ has exactly N sides, i.e., none of any three adjacent points are collinear. We say \mathcal{P} is a convex polygon if it is the boundary of a convex set. Without loss of generality, we assume that Y_j is arranged in an anticlockwise way. We define the oriented area of three points $Y_1, Y_2, Y_3 \in \mathbb{R}^2$ as

$$\text{Area}(Y_1, Y_2, Y_3) := \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2} (Y_3 - Y_2) \cdot (Y_2 - Y_1)^\perp,$$

where $Y_i = (x_i, y_i)$, $i = 1, 2, 3$. The following characterizations of convexity are straightforward.

LEMMA 3.1. *Let $\mathcal{P} = (X_1, \dots, X_N)$ be an N -polygon. The following statements are equivalent:*

- (i) *The N -polygon $\mathcal{P} = (X_1, \dots, X_N)$ is convex.*
- (ii) *Any internal angle $\angle X_{j-1}X_jX_{j+1} < \pi$ for $j = 1, \dots, N$.*
- (iii) *$S_j := \text{Area}(X_{j-1}, X_j, X_{j+1}) > 0$ for $j = 1, \dots, N$.*
- (iv) *$S_j^k := \text{Area}(X_{j-1}, X_j, X_k) > 0$ for $j = 1, \dots, N$ and $k \neq j - 1, j$.*

Here, we set $X_0 = X_N$ and $X_{N+1} = X_1$ when involved.

Proof. Clearly we have (i) \Leftrightarrow (ii) \Leftrightarrow (iii) and (iv) \Rightarrow (iii). It suffices to show (i) \Rightarrow (iv). Indeed, by the support property of convex polygons [24, Theorem 4.2], for any $X \in \overline{X_{j-1}X_j}$, there exists a support hyperplane $\ell(X)$ such that \mathcal{P} is contained in one of the two closed halfspaces determined by $\ell(X)$. It is obvious that $\overline{X_{j-1}X_j} \subseteq \ell(X)$. In particular, for any $k \neq j - 1, j$, X_k lies in the same halfspace determined by $\ell(X)$, i.e., there exists a nonzero vector $\mathcal{N} \in \mathbb{R}^2$ such that

$$(3.7) \quad \mathcal{N} \cdot (X_j - X_{j-1}) = 0, \quad \mathcal{N} \cdot (X_k - X_{j-1}) > 0 \quad (k \neq j - 1, j).$$

Thus we can write $\mathcal{N} = \varepsilon(X_j - X_{j-1})^\perp$. Noticing that

$$2S_j^k = (X_k - X_j) \cdot (X_j - X_{j-1})^\perp = (X_k - X_{j-1}) \cdot \mathcal{N} / \varepsilon, \quad k \neq j - 1, j,$$

which together with (iii) implies $\varepsilon > 0$, this yields (iv) by recalling (3.7), and the proof is completed. \square

Inspired by (2.3) and (3.1), to prove Theorem 2.3, it suffices to show that for any $t > 0$, the N -polygon $\mathcal{P} = (X_1, \dots, X_N)$ is convex. We first compute the evolution formula of the oriented area of the triangles consisting of three adjacent vertices.

LEMMA 3.2. *Under the flow (3.1), if $q_j > 0$ for any j , then the oriented area $S_j(t)$ satisfies*

$$(3.8) \quad \begin{aligned} \frac{d}{dt} S_j(t) &= -a_j \cdot S_j + b_j \cdot S_{j+1}^{j-2} + c_j \cdot S_j^{j+2} + \frac{1}{q_j + q_{j+1}} \frac{\pi}{L_h} |X_{j+1} - X_{j-1}|^2 \\ &+ \frac{1}{q_{j-1} + q_j} \frac{\pi}{L_h} (X_j - X_{j+1}) \cdot (X_j - X_{j-2}) \\ &+ \frac{1}{q_{j+1} + q_{j+2}} \frac{\pi}{L_h} (X_j - X_{j-1}) \cdot (X_j - X_{j+2}), \end{aligned}$$

where a_j, b_j, c_j are positive functions defined by

$$a_j = \frac{2}{q_j q_{j-1}} + \frac{2}{q_j q_{j+1}} + \frac{2}{q_{j+1} q_{j+2}}, \quad b_j = \frac{2}{q_{j-1}(q_{j-1} + q_j)}, \quad c_j = \frac{2}{q_{j+2}(q_{j+1} + q_{j+2})}.$$

Proof. By definition, it can be observed that

$$S_j = \frac{1}{2} (X_{j+1} - X_j) \cdot (X_j - X_{j-1})^\perp = \frac{q_j q_{j+1}}{2} \mathcal{T}_{j+1} \cdot \mathcal{N}_j.$$

Employing the flow equation (3.1), we derive

$$\begin{aligned} \frac{d}{dt} S_j &= \frac{1}{2} \frac{d}{dt} (X_{j+1} \cdot X_j^\perp + X_j \cdot X_{j-1}^\perp - X_{j+1} \cdot X_{j-1}^\perp) \\ &= \frac{1}{2} \frac{dX_{j+1}}{dt} \cdot (X_j^\perp - X_{j-1}^\perp) - \frac{1}{2} \frac{dX_j}{dt} \cdot (X_{j+1}^\perp - X_{j-1}^\perp) + \frac{1}{2} \frac{dX_{j-1}}{dt} \cdot (X_{j+1}^\perp - X_j^\perp) \\ &\triangleq: J_1 + J_2 + J_3, \end{aligned}$$

where we have used the property $u \cdot v^\perp = -v \cdot u^\perp$ for any $u, v \in \mathbb{R}^2$. Applying (3.1), one can calculate J_j as

$$\begin{aligned} (q_{j+1} + q_{j+2}) J_1 &= \frac{q_j}{2} (q_{j+1} + q_{j+2}) \dot{X}_{j+1} \cdot \mathcal{N}_j \\ &= q_j \left(\mathcal{T}_{j+2} - \mathcal{T}_{j+1} - \frac{\pi}{L_h} (X_{j+2} - X_j)^\perp \right) \cdot \mathcal{N}_j \\ &= \frac{X_{j+2} - X_j}{q_{j+2}} \cdot q_j \mathcal{N}_j - \left(1 + \frac{q_{j+1}}{q_{j+2}} \right) \mathcal{T}_{j+1} \cdot q_j \mathcal{N}_j - \frac{\pi}{L_h} (X_{j+2} - X_j) \cdot q_j \mathcal{T}_j \\ &= \frac{2}{q_{j+2}} S_j^{j+2} - \left(\frac{2}{q_{j+1}} + \frac{2}{q_{j+2}} \right) S_j - \frac{\pi}{L_h} (X_j - X_{j-1}) \cdot (X_{j+2} - X_j). \end{aligned}$$

Similarly one easily gets

$$\begin{aligned} (q_j + q_{j+1}) J_2 &= - \left(\frac{2}{q_j} + \frac{2}{q_{j+1}} \right) S_j + \frac{\pi}{L_h} |X_{j+1} - X_{j-1}|^2, \\ (q_{j-1} + q_j) J_3 &= - \left(\frac{2}{q_{j-1}} + \frac{2}{q_j} \right) S_j + \frac{2}{q_{j-1}} S_{j+1}^{j-2} + \frac{\pi}{L_h} (X_j - X_{j+1}) \cdot (X_j - X_{j-2}). \end{aligned}$$

Combining the above equations together yields (3.8) immediately. \square

Now we turn to the proof of convexity-preservation.

Proof of Theorem 2.3. Define the function $F(t) := \min\{S_j(t), 1 \leq j \leq N\}$. It follows from the assumption that $F(0) > 0$. By the definition of N -polygon and Lemma 3.1(iii), it suffices to show $F(t) > 0$ for $t \in (0, T]$.

We argue by contradiction. Suppose the contrary; by continuity there exists the smallest time $0 < t_0 \leq T$ such that $F(t_0) = 0$. Then we have the following:

- (1) there exists some triangle such that the oriented area achieves zero at t_0 ; without loss of generality, we may assume $S_2(t_0) = \text{Area}(X_1, X_2, X_3)(t_0) = 0$;
- (2) the N -polygon $\mathcal{P}(t) = (X_1(t), \dots, X_N(t))$ is convex for $0 \leq t < t_0$; hence by Lemma 3.1(iv), it holds that

$$S_j^k(t) > 0, \quad \forall 0 \leq t < t_0, \quad \forall j = 1, \dots, N, \quad \forall k \neq j - 1, j.$$

Thus by (3.8), one has

$$(3.9) \quad 0 \geq \frac{d}{dt} S_2(t_0) = b_2 \cdot S_3^0(t_0) + c_2 \cdot S_2^4(t_0) + Q(t_0),$$

where

$$Q(t_0) = \frac{1}{q_2 + q_3} \frac{\pi}{L_h} |X_3 - X_1|^2(t_0) + \frac{1}{q_1 + q_2} \frac{\pi}{L_h} (X_2 - X_3) \cdot (X_2 - X_0)(t_0) + \frac{1}{q_3 + q_4} \frac{\pi}{L_h} (X_2 - X_1) \cdot (X_2 - X_4)(t_0).$$

Notice that (1) implies that $X_1(t_0), X_2(t_0), X_3(t_0)$ are collinear. There are two possibilities: (i) $(X_2 - X_1) \cdot (X_3 - X_2)(t_0) > 0$; (ii) $(X_2 - X_1) \cdot (X_3 - X_2)(t_0) < 0$. Next we discuss it case by case.

Case (i). $(X_2 - X_1) \cdot (X_3 - X_2)(t_0) > 0$. First by (2) and continuity, we easily find that $S_3^0(t_0) \geq 0, S_2^4(t_0) \geq 0$. We claim that $Q(t_0) \geq 0$. Actually, notice that in this case it holds that

$$|X_3 - X_1|(t_0) = |X_3 - X_2|(t_0) + |X_2 - X_1|(t_0) = q_2(t_0) + q_3(t_0),$$

which implies

$$\frac{\pi}{L_h} \frac{|X_3 - X_1|^2}{q_2 + q_3}(t_0) = \frac{\pi}{L_h} \frac{(q_2 + q_3)^2}{q_2 + q_3}(t_0) = \frac{\pi}{L_h} (q_2 + q_3)(t_0).$$

On the other hand, by the triangle inequality, one can estimate

$$\frac{1}{q_1 + q_2} \frac{\pi}{L_h} (X_2 - X_3) \cdot (X_2 - X_0) \geq -\frac{1}{q_1 + q_2} \frac{\pi}{L_h} (q_3 \cdot (q_1 + q_2)) = -\frac{\pi}{L_h} q_3,$$

$$\frac{1}{q_3 + q_4} \frac{\pi}{L_h} (X_2 - X_1) \cdot (X_2 - X_4) \geq -\frac{\pi}{L_h} q_2.$$

Thus

$$Q(t_0) \geq \frac{\pi}{L_h} (q_2 + q_3)(t_0) - \frac{\pi}{L_h} q_3(t_0) - \frac{\pi}{L_h} q_2(t_0) = 0.$$

Recalling (3.9), all the above inequalities become equalities, i.e., $S_3^0(t_0) = S_2^4(t_0) = 0$, and

$$|X_2 - X_0|(t_0) = q_1(t_0) + q_2(t_0), \quad |X_2 - X_4|(t_0) = q_2(t_0) + q_3(t_0).$$

This means X_0, X_1, X_2, X_3, X_4 are collinear at t_0 and are arranged in order, i.e., $(X_{j+1} - X_j)(t_0) = d_j(X_1 - X_0)(t_0)$ with $d_j > 0, j = 1, 2, 3$. In particular,

$$S_3(t_0) = 0, \quad (X_3 - X_2) \cdot (X_4 - X_3)(t_0) > 0.$$

Repeating the above procedure another $N-4$ times, we get that at time t_0 , all vertices (X_0, X_1, \dots, X_N) are collinear and are arranged in order, i.e.,

$$X_{j+1}(t_0) - X_j(t_0) = d_j(X_1(t_0) - X_0(t_0)), \quad d_j > 0, \quad j = 1, 2, \dots, N-1,$$

which contradicts the periodic condition $X_0 = X_N$.

Case (ii). $(X_2 - X_1) \cdot (X_3 - X_2)(t_0) < 0$. This means $(X_3 - X_2)(t_0) = d_2(X_2 - X_1)(t_0)$ with $d_2 < 0$. By (2) and continuity, we have $S_3(t_0) \geq 0$, $S_2^4(t_0) \geq 0$. On the other hand, by definition, one finds

$$\begin{aligned} S_3(t_0) &= \frac{1}{2}(X_4 - X_3) \cdot (X_3 - X_2)^\perp(t_0) = \frac{1}{2}(X_4 - X_2) \cdot (X_3 - X_2)^\perp(t_0) \\ &= \frac{d_2}{2}(X_4 - X_2) \cdot (X_2 - X_1)^\perp(t_0) = d_2 S_2^4(t_0) \leq 0. \end{aligned}$$

It follows that $S_3(t_0) = 0$ and X_1, X_2, X_3, X_4 are collinear. We claim that

$$(X_4 - X_3) \cdot (X_3 - X_2)(t_0) < 0.$$

Otherwise, Case (i) happens for the collinear points X_2, X_3, X_4 . Differentiating S_3 at t_0 and repeating the arguments as in Case (i) involving S_3 will lead to the conclusion that X_1, X_2, X_3, X_4, X_5 are collinear and are arranged in order, which contradicts the premise that $(X_2 - X_1) \cdot (X_3 - X_2)(t_0) < 0$. Thus it holds that

$$S_3(t_0) = 0, \quad (X_4 - X_3) \cdot (X_3 - X_2)(t_0) < 0.$$

Repeating this argument, we can conclude that all vertices X_1, X_2, \dots, X_N are collinear; furthermore, every three adjacent vertices are interlaced, i.e.,

$$(X_{j+1} - X_j) \cdot (X_j - X_{j-1})(t_0) < 0, \quad j = 1, \dots, N.$$

In particular, all exterior angles of the polygon $\mathcal{P}(t_0)$ are π . On the other hand, noticing each exterior angle α_j is continuous, by continuity and convexity, we have

$$N\pi = \sum_{j=1}^N \alpha_j(t_0) = \lim_{t \rightarrow t_0} \sum_{j=1}^N \alpha_j(t) = 2\pi,$$

which leads to a contradiction since $N \geq 3$. \square

Remark 3.1. A similar argument holds for Dziuk's semidiscrete scheme [19] for the CSF. More precisely, under nondegeneration of vertices ($q_j > 0$) we can first prove that if the initial polygon is convex, then the evolved polygon under the semidiscrete scheme of the CSF is also convex unless all vertices are collinear at some $t_0 > 0$, in which case the area vanishes, i.e., $\text{Area}(\mathcal{P}(t_0)) = 0$. On the other hand, applying the error estimate of the scheme for the CSF [19, 20], we arrive at

$$\begin{aligned} |\text{Area}(\Gamma_{t_0}) - \text{Area}(\mathcal{P}(t_0))| &= \left| \int_{\mathbb{S}^1} \partial_\xi x \cdot y \, d\xi - \int_{\mathbb{S}^1} \partial_\xi x_h \cdot y_h \, d\xi \right| \\ &\leq \int_{\mathbb{S}^1} |\partial_\xi x - \partial_\xi x_h| \cdot |y| \, d\xi + \int_{\mathbb{S}^1} |\partial_\xi x_h| \cdot |y - y_h| \, d\xi \\ &\leq C \sup_{[0, T]} \|X - X_h\|_{H^1(\mathbb{S}^1)} \leq Ch, \end{aligned}$$

where Γ_t represents the real curve driven by the CSF. This implies that $\text{Area}(\mathcal{P}(t_0))$ stays away from zero if $\text{Area}(\Gamma_t)$ has a positive lower bound for $t \in (0, T]$ and h is small enough. This leads to a contradiction!

3.2. Proof of Theorem 2.4.

Proof. Applying (3.3), we get the derivative of the perimeter

$$\begin{aligned} \frac{d}{dt}L_h &= \sum_{j=1}^N \frac{d}{dt}q_j = \sum_{j=1}^N \left(-\frac{1}{q_j + q_{j+1}} |\mathcal{T}_{j+1} - \mathcal{T}_j|^2 - \frac{1}{q_j + q_{j-1}} |\mathcal{T}_{j-1} - \mathcal{T}_j|^2 \right) \\ &\quad + \frac{2\pi}{L_h} \sum_{j=1}^N \left(-\frac{q_{j+1}}{q_j + q_{j+1}} \mathcal{T}_j \cdot \mathcal{N}_{j+1} + \frac{q_{j-1}}{q_j + q_{j-1}} \mathcal{T}_j \cdot \mathcal{N}_{j-1} \right) \\ &= -2 \sum_{j=1}^N \frac{|\mathcal{T}_{j+1} - \mathcal{T}_j|^2}{q_j + q_{j+1}} - \frac{2\pi}{L_h} \sum_{j=1}^N \mathcal{T}_j \cdot \mathcal{N}_{j+1}, \end{aligned}$$

where we have used the fact that

$$\begin{aligned} &\sum_{j=1}^N \left(-\frac{q_{j+1}}{q_j + q_{j+1}} \mathcal{T}_j \cdot \mathcal{N}_{j+1} + \frac{q_{j-1}}{q_j + q_{j-1}} \mathcal{T}_j \cdot \mathcal{N}_{j-1} \right) \\ &= \sum_{j=1}^N \left(-\frac{q_{j+1}}{q_j + q_{j+1}} \mathcal{T}_j \cdot \mathcal{N}_{j+1} - \frac{q_j}{q_j + q_{j+1}} \mathcal{N}_{j+1} \cdot \mathcal{T}_j \right) = -\sum_{j=1}^N \mathcal{T}_j \cdot \mathcal{N}_{j+1}. \end{aligned}$$

We denote α_j by the exterior angle of the polygon at X_j . By Theorem 2.3, $\mathcal{P}(t)$ remains convex for all t , which implies that $0 < \alpha_j < \pi$ for $j = 1, \dots, N$, and $\sum_{j=1}^N \alpha_j = 2\pi$. Direct computations yield

$$|\mathcal{T}_{j+1} - \mathcal{T}_j|^2 = 2 - 2 \cos \alpha_j = 4 \sin^2(\alpha_j/2) \quad \text{and} \quad \mathcal{T}_j \cdot \mathcal{N}_{j+1} = -\sin \alpha_j.$$

By the Cauchy–Schwarz inequality, one easily gets

$$\sum_{j=1}^N 2 \sin \left(\frac{\alpha_j}{2} \right) \leq \left(\sum_{j=1}^N \frac{4 \sin^2 \left(\frac{\alpha_j}{2} \right)}{q_j + q_{j+1}} \right)^{\frac{1}{2}} \left(\sum_{j=1}^N q_j + q_{j+1} \right)^{\frac{1}{2}} = (2L_h)^{\frac{1}{2}} \left(\sum_{j=1}^N \frac{4 \sin^2 \left(\frac{\alpha_j}{2} \right)}{q_j + q_{j+1}} \right)^{\frac{1}{2}}.$$

Hence we derive

$$\frac{dL_h}{dt} = -2 \sum_{j=1}^N \left(\frac{4 \sin^2 \left(\frac{\alpha_j}{2} \right)}{q_j + q_{j+1}} - \frac{\pi}{L_h} \sin \alpha_j \right) \leq \frac{-2}{L_h} \left(2 \left(\sum_{j=1}^N \sin \left(\frac{\alpha_j}{2} \right) \right)^2 - \pi \sum_{j=1}^N \sin \alpha_j \right) \leq 0,$$

where in the last inequality we have utilized a trigonometric inequality (cf. Lemma 3.3 below) and the proof is completed. \square

LEMMA 3.3. *Define*

$$f_N(\beta_1, \dots, \beta_N) := \left(\sum_{j=1}^N \sin \beta_j \right)^2 - \frac{1}{2} \left(\sum_{j=1}^N \beta_j \right) \left(\sum_{j=1}^N \sin(2\beta_j) \right), \quad 0 \leq \beta_j \leq \frac{\pi}{2}.$$

Then it holds that $f_N(\beta_1, \dots, \beta_N) \geq 0$.

Proof. We prove $f_N(\beta_1, \dots, \beta_N) \geq 0$ by induction. For $N = 1$, we have

$$f_1(\beta_1) = \sin^2 \beta_1 - \frac{\beta_1}{2} \cdot \sin(2\beta_1) = \sin \beta_1 \cos \beta_1 (\tan \beta_1 - \beta_1) \geq 0.$$

Now suppose $f_{N-1}(\beta_1, \dots, \beta_{N-1}) \geq 0$; we first compute

$$\begin{aligned} \frac{\partial f_N(\beta_1, \dots, \beta_N)}{\partial \beta_N} &= 2 \cos \beta_N \cdot \sum_{j=1}^N \sin \beta_j - \frac{1}{2} \sum_{j=1}^N \sin(2\beta_j) - \sum_{j=1}^N \beta_j \cdot \cos(2\beta_N) \\ &\geq 2 \cos^2 \beta_N \cdot \sum_{j=1}^N \sin \beta_j - \sum_{j=1}^N \sin \beta_j \cos \beta_j - \sum_{j=1}^N \beta_j \cdot \cos(2\beta_N) \\ &= \sum_{j=1}^N (\sin \beta_j - \sin \beta_j \cos \beta_j - (\beta_j - \sin \beta_j) \cdot \cos(2\beta_N)) \\ &\geq \sum_{j=1}^N (\sin \beta_j - \sin \beta_j \cos \beta_j - (\beta_j - \sin \beta_j)) =: \sum_{j=1}^N B_j(\beta_j). \end{aligned}$$

Notice that

$$\frac{\partial B_j}{\partial \beta_j} = \cos \beta_j - \cos^2 \beta_j + \sin^2 \beta_j - (1 - \cos \beta_j) = 2 \cos \beta_j - 2 \cos^2 \beta_j \geq 0;$$

this implies that B_j is increasing and, particularly,

$$B_j(\beta_j) \geq B_j(0) = 0, \quad \beta_j \in [0, \pi/2], \quad j = 1, \dots, N.$$

Hence one gets $\frac{\partial f_N(\beta_1, \dots, \beta_N)}{\partial \beta_N} \geq 0$, and by induction,

$$f_N(\beta_1, \dots, \beta_N) \geq f_N(\beta_1, \dots, \beta_{N-1}, 0) = f_{N-1}(\beta_1, \dots, \beta_{N-1}) \geq 0,$$

which completes the proof. \square

4. Proof of Theorem 2.5. In this section, we present the error estimate by following the lines of Dziuk's argument [20] and Pozzi and Stinner's computation [39]. We establish the stability estimate and length element difference under the assumption of boundedness of the semidiscrete length element. Then a bound of the semidiscrete length element is given. All above preliminary estimates together with the continuity argument enable us to derive the desired error bound. Throughout this section, we assume that Assumptions 2.1 and 2.2 are always valid, and we denote $C > 0$ by a general constant that may change from line to line. For simplicity, we omit the space whenever the norm is defined on \mathbb{S}^1 .

We first give the stability estimate.

LEMMA 4.1. *Assume further that the solution of (2.4) satisfies*

$$(4.1) \quad \inf_{\xi} |\partial_{\xi} X_h| \geq c_0 > 0 \quad \text{and} \quad \sup_{\xi} |\partial_{\xi} X_h| \leq C_0 \quad \forall 0 \leq t \leq T^* \leq T.$$

Then for any $t \in [0, T^*]$, we have

$$(4.2) \quad \int_0^t \int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h d\xi ds + \sup_{0 \leq s \leq t} \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h d\xi \leq C \int_0^t \|q - q_h\|_{L^2}^2 ds + Ch^2,$$

where $\mathcal{T} = \frac{\partial_{\xi} X}{|\partial_{\xi} X|}$, $\mathcal{T}_h = \frac{\partial_{\xi} X_h}{|\partial_{\xi} X_h|}$, $q = |\partial_{\xi} X|$, $q_h = |\partial_{\xi} X_h|$, and C depends on C_p , C_P , κ_1 , c_0 , C_0 , and $K(X)$.

Proof. We first notice that the boundedness of the length element will imply the boundedness of the perimeter. Indeed, by Assumption 2.2 and (4.1), one easily gets

$$(4.3) \quad 2\pi\kappa_1 \leq L \leq 2\pi\kappa_2, \quad 2\pi c_0 \leq L_h \leq 2\pi C_0 \quad \forall t \in [0, T^*].$$

Recalling $\partial_\xi X \neq 0, \partial_\xi X_h \neq 0, |\mathcal{T}| = |\mathcal{T}_h| = 1$, this enables us to write the following:

$$(4.4) \quad \begin{aligned} |\partial_\xi X - \partial_\xi X_h|^2 &= |\partial_\xi X|^2 + |\partial_\xi X_h|^2 - 2\partial_\xi X \cdot \partial_\xi X_h \\ &= (q - q_h)^2 + 2qq_h - 2qq_h\mathcal{T} \cdot \mathcal{T}_h \\ &= (q - q_h)^2 + qq_h(2 - 2\mathcal{T} \cdot \mathcal{T}_h) = (q - q_h)^2 + qq_h|\mathcal{T} - \mathcal{T}_h|^2. \end{aligned}$$

Taking the difference between (1.4) and (2.4), we obtain that the error equation

$$\begin{aligned} &\int_{\mathbb{S}^1} (|\partial_\xi X| |\partial_t X - \partial_t X_h| - |\partial_\xi X_h| |\partial_t X_h|) \cdot v_h \, d\xi + \int_{\mathbb{S}^1} (\mathcal{T} - \mathcal{T}_h) \cdot \partial_\xi v_h \, d\xi \\ &+ \int_{\mathbb{S}^1} \left(\frac{2\pi}{L} (\partial_\xi X)^\perp - \frac{2\pi}{L_h} (\partial_\xi X_h)^\perp \right) \cdot v_h \, d\xi = \int_{\mathbb{S}^1} \frac{\mathbf{h}^2 |\partial_\xi X_h|}{6} \partial_\xi \partial_t X_h \cdot \partial_\xi v_h \, d\xi \end{aligned}$$

holds for any $v_h \in V_h$. Taking $v_h = I_h(\partial_t X) - \partial_t X_h \in V_h$ in the above equation yields

$$\begin{aligned} &\int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h \, d\xi + \int_{\mathbb{S}^1} (\mathcal{T} - \mathcal{T}_h) (\partial_\xi \partial_t X - \partial_\xi \partial_t X_h) \, d\xi \\ &= \int_{\mathbb{S}^1} \partial_t X \cdot (q_h - q) (I_h \partial_t X - \partial_t X_h) \, d\xi + \int_{\mathbb{S}^1} \frac{\mathbf{h}^2 q_h}{6} \partial_\xi \partial_t X_h \cdot \partial_\xi (I_h \partial_t X - \partial_t X_h) \, d\xi \\ &+ \int_{\mathbb{S}^1} q_h \cdot (\partial_t X - \partial_t X_h) (\partial_t X - I_h \partial_t X) \, d\xi + \int_{\mathbb{S}^1} (\mathcal{T} - \mathcal{T}_h) \cdot (\partial_\xi \partial_t X - \partial_\xi I_h \partial_t X) \, d\xi \\ &+ \int_{\mathbb{S}^1} \frac{2\pi}{L} (\partial_\xi X - \partial_\xi X_h)^\perp \cdot (\partial_t X_h - I_h \partial_t X) \, d\xi \\ &+ \int_{\mathbb{S}^1} \left(\frac{2\pi}{L} - \frac{2\pi}{L_h} \right) (\partial_\xi X_h)^\perp \cdot (\partial_t X_h - I_h \partial_t X) \, d\xi \triangleq: J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned}$$

The estimates of the second term on the left side and J_j for $1 \leq j \leq 4$ can be found in [20, Lemma 5.1] and read as

$$\begin{aligned} &\int_{\mathbb{S}^1} (\mathcal{T} - \mathcal{T}_h) \cdot (\partial_\xi \partial_t X - \partial_\xi \partial_t X_h) \, d\xi \\ &\geq \frac{d}{dt} \left(\int_{\mathbb{S}^1} (1 - \mathcal{T} \cdot \mathcal{T}_h) q_h \, d\xi \right) - C \|\partial_\xi \partial_t X\|_{L^\infty} \left(\int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h \, d\xi + \|q - q_h\|_{L^2}^2 \right), \\ J_1 &\leq \varepsilon \int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h \, d\xi + C(\varepsilon) \|\partial_t X\|_{L^\infty}^2 \int_{\mathbb{S}^1} \frac{(q - q_h)^2}{q_h} \, d\xi + C \|q_h\|_{L^\infty}^2 \|\partial_t X\|_{H^1}^2 h^2 \\ &\leq \varepsilon \int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h \, d\xi + C(\varepsilon) \|\partial_t X\|_{L^\infty}^2 \|q - q_h\|_{L^2}^2 + Ch^2 \|\partial_t X\|_{H^1}^2, \\ J_2 &\leq \frac{1}{24} \|q_h\|_{L^\infty} \|\partial_t X\|_{H^1}^2 h^2 \leq Ch^2 \|\partial_t X\|_{H^1}^2, \\ J_3 &\leq \varepsilon \int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h \, d\xi + C(\varepsilon) \|q_h\|_{L^\infty} \|\partial_t X\|_{H^1}^2 h^2 \\ &\leq \varepsilon \int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h \, d\xi + C(\varepsilon) h^2 \|\partial_t X\|_{H^1}^2, \\ J_4 &\leq C \|\partial_t X\|_{H^2} \|\mathcal{T} - \mathcal{T}_h\|_{L^2} h \leq C \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h \, d\xi + Ch^2 \|\partial_t X\|_{H^2}^2, \end{aligned}$$

where ε is a generic small positive constant which will be chosen later. It remains to estimate J_5 and J_6 . For J_5 , we decompose it as $J_5 = J_{51} + J_{52}$ with

$$\begin{aligned} J_{51} &= \int_{\mathbb{S}^1} \frac{2\pi}{L} (\partial_\xi X - \partial_\xi X_h)^\perp \cdot (\partial_t X - I_h \partial_t X) \, d\xi, \\ J_{52} &= \int_{\mathbb{S}^1} \frac{2\pi}{L} (\partial_\xi X - \partial_\xi X_h)^\perp \cdot (\partial_t X_h - \partial_t X) \, d\xi. \end{aligned}$$

Applying Assumption 2.2, (4.3), (4.4), and the interpolation estimate (2.1), we derive

$$\begin{aligned} J_{51} &\leq C \int_{\mathbb{S}^1} |\partial_\xi X - \partial_\xi X_h|^2 \, d\xi + C \int_{\mathbb{S}^1} |\partial_t X - I_h \partial_t X|^2 \, d\xi \\ &= C \left(\int_{\mathbb{S}^1} qq_h (|\mathcal{T} - \mathcal{T}_h|^2 + (q - q_h)^2) \, d\xi + \|\partial_t X - I_h \partial_t X\|_{L^2}^2 \right) \\ &\leq C \|\partial_\xi X\|_{L^\infty} \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h \, d\xi + C \int_{\mathbb{S}^1} (q - q_h)^2 \, d\xi + Ch^2 \|\partial_t X\|_{H^1}^2, \\ J_{52} &\leq C(\varepsilon) \int_{\mathbb{S}^1} |\partial_\xi X - \partial_\xi X_h|^2 \, d\xi + \varepsilon \int_{\mathbb{S}^1} |\partial_t X_h - \partial_t X|^2 q_h \, d\xi \\ &= C(\varepsilon) \int_{\mathbb{S}^1} qq_h |\mathcal{T} - \mathcal{T}_h|^2 + (q - q_h)^2 \, d\xi + \varepsilon \int_{\mathbb{S}^1} |\partial_t X_h - \partial_t X|^2 q_h \, d\xi \\ &\leq C(\varepsilon) \|\partial_\xi X\|_{L^\infty} \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h \, d\xi + C(\varepsilon) \|q - q_h\|_{L^2}^2 + \varepsilon \int_{\mathbb{S}^1} |\partial_t X_h - \partial_t X|^2 q_h \, d\xi. \end{aligned}$$

Similarly we decompose $J_6 = J_{61} + J_{62}$ with

$$\begin{aligned} J_{61} &= \int_{\mathbb{S}^1} \left(\frac{2\pi}{L} - \frac{2\pi}{L_h} \right) (\partial_\xi X_h)^\perp \cdot (\partial_t X - I_h \partial_t X) \, d\xi, \\ J_{62} &= \int_{\mathbb{S}^1} \left(\frac{2\pi}{L} - \frac{2\pi}{L_h} \right) (\partial_\xi X_h)^\perp \cdot (\partial_t X_h - \partial_t X) \, d\xi. \end{aligned}$$

Notice that

$$(4.5) \quad |L - L_h| \leq \|q - q_h\|_{L^1} \leq C \|q - q_h\|_{L^2};$$

this together with (4.3), (4.1), and (2.1) leads to

$$\begin{aligned} J_{61} &\leq C |L - L_h|^2 + C \|\partial_t X - I_h \partial_t X\|_{L^2}^2 \leq C \|q - q_h\|_{L^2}^2 + Ch^2 \|\partial_t X\|_{H^1}^2, \\ J_{62} &\leq C \int_{\mathbb{S}^1} |L_h - L| q_h^{\frac{1}{2}} |\partial_t X_h - \partial_t X| \, d\xi \leq C(\varepsilon) |L_h - L|^2 + \varepsilon \int_{\mathbb{S}^1} q_h |\partial_t X_h - \partial_t X|^2 \, d\xi \\ &\leq C(\varepsilon) \|q - q_h\|_{L^2}^2 + \varepsilon \int_{\mathbb{S}^1} q_h |\partial_t X_h - \partial_t X|^2 \, d\xi. \end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned} &\int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h \, d\xi + \frac{d}{dt} \int_{\mathbb{S}^1} (1 - \mathcal{T} \cdot \mathcal{T}_h) q_h \, d\xi \leq 4\varepsilon \int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h \, d\xi \\ &\quad + C(\varepsilon) h^2 \|\partial_t X\|_{H^2}^2 + C(\varepsilon, K(X)) \|q - q_h\|_{L^2}^2 + C(\varepsilon, K(X)) \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h \, d\xi. \end{aligned}$$

Choosing ε small enough, integrating both sides with respect to time from 0 to t , noticing that

$$\begin{aligned} & \int_{\mathbb{S}^1} (1 - \mathcal{T} \cdot \mathcal{T}_h)(0)q_h(0)d\xi = \frac{1}{2} \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2(0)q_h(0)d\xi \\ & \leq C \int_{\mathbb{S}^1} |\partial_\xi X - \partial_\xi X_h|^2(0)d\xi \leq C \|\partial_\xi(X - I_h X)(0)\|_{L^2}^2 \leq Ch^2 \|X^0\|_{H^2}^2, \end{aligned}$$

we are led to the estimate (4.2) with appropriate constant C by applying Gronwall's inequality and Sobolev embedding $H^1(\mathbb{S}^1) \hookrightarrow L^\infty(\mathbb{S}^1)$. \square

LEMMA 4.2. *Suppose*

$$\int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h d\xi + \|q - q_h\|_{L^2}^2 \leq C_1 h^2 \quad \forall t \in [0, T^*];$$

then there exists a constant h_0 such that for any $0 < h \leq h_0$, we have

$$\inf_{\xi} q_h \geq 3\kappa_1/4 \quad \text{and} \quad \sup_{\xi} q_h \leq 3\kappa_2/2 \quad \forall t \in [0, T^*],$$

where the constant h_0 depends on $C_1, C_p, C_P, \kappa_1, \kappa_2$, and $K(X)$.

Proof. Applying the triangle inequality, the interpolation error estimate (2.1), and the inverse estimate (2.2), we can derive

$$\begin{aligned} \|\partial_\xi X - \partial_\xi X_h\|_{L^\infty} & \leq \|\partial_\xi X - I_h \partial_\xi X\|_{L^\infty} + \|\partial_\xi X_h - I_h \partial_\xi X\|_{L^\infty} \\ & \leq Ch^{1/2} \|\partial_\xi X\|_{H^1} + Ch^{-1/2} \|\partial_\xi X_h - I_h \partial_\xi X\|_{L^2} \\ & \leq Ch^{1/2} \|X\|_{H^2} + Ch^{-1/2} \|\partial_\xi X_h - \partial_\xi X\|_{L^2}. \end{aligned}$$

The assumption and equality (4.4) imply

$$\|\partial_\xi X - \partial_\xi X_h\|_{L^2} \leq \left(\int_{\mathbb{S}^1} qq_h |\mathcal{T} - \mathcal{T}_h|^2 d\xi \right)^{1/2} + \|q - q_h\|_{L^2} \leq \sqrt{C_1} (1 + \|\partial_\xi X\|_{L^\infty}^{1/2}) h.$$

Then it follows that

$$\|\partial_\xi X - \partial_\xi X_h\|_{L^\infty} \leq C(K(X), C_1) h^{1/2},$$

and the conclusion follows by recalling Assumption 2.2. \square

In order to estimate the length element difference, we first give the following preliminary lemma by following the lines of [39, Lemmas 3.2 and 4.1].

LEMMA 4.3. *Under the assumptions of Lemma 4.1, there exists a constant C depending on $\kappa_1, C_p, C_P, c_0, C_0, T$ such that the following estimates hold for $t \in [0, T^*]$:*

$$(4.6) \quad |R - R_j| \leq C(|L - L_h| + |\mathcal{T} - \mathcal{T}_j| + |\mathcal{T} - \mathcal{T}_{j+1}|), \quad j = 1, \dots, N,$$

$$(4.7) \quad \int_0^t (q_j + q_{j+1}) |\dot{X}_j - R_j|^2 ds \leq Ch, \quad j = 1, \dots, N,$$

where R and R_j are defined as in (3.5). Moreover, we have the estimates on I_j :

$$(4.8) \quad \begin{aligned} & \sum_{k=j-1}^{j+1} \|\mathcal{T} - \mathcal{T}_k\|_{L^2(I_j)}^2 \leq Ch^2 \|X\|_{H^2(S_j)}^2 + C \|\mathcal{T} - \mathcal{T}_h\|_{L^2(S_j)}^2, \\ & \sum_{k=j-1}^j \|\partial_t X - \dot{X}_k\|_{L^2(I_j)}^2 \leq Ch^2 \|\partial_t X\|_{H^1(I_j)}^2 + C \|\partial_t X - \partial_t X_h\|_{L^2(I_j)}^2, \\ & \sum_{k=j-1}^{j+1} \|qh_j - q_k\|_{L^2(I_j)}^2 \leq Ch^4 \|X\|_{H^2(S_j)}^2 + Ch^2 \|q - q_h\|_{L^2(S_j)}^2, \end{aligned}$$

with $S_j = I_j \cup I_{j+1} \cup I_{j-1}$.

Proof. First, (4.6) can be easily derived by employing Assumption 2.1 and (4.1):

$$\begin{aligned} |R - R_j| &= \left| -\frac{2\pi}{L}\mathcal{N} + \frac{2\pi}{L_h} \frac{\mathcal{N}_j q_j + \mathcal{N}_{j+1} q_{j+1}}{q_j + q_{j+1}} \right| \\ &\leq \left| -\frac{2\pi}{L}\mathcal{N} + \frac{2\pi}{L_h}\mathcal{N} \right| + \left| \frac{2\pi}{L_h} \frac{q_j(\mathcal{N} - \mathcal{N}_j)}{q_j + q_{j+1}} \right| + \left| \frac{2\pi}{L_h} \frac{q_{j+1}(\mathcal{N} - \mathcal{N}_{j+1})}{q_j + q_{j+1}} \right| \\ &\leq C(|L - L_h| + |\mathcal{T} - \mathcal{T}_j| + |\mathcal{T} - \mathcal{T}_{j+1}|). \end{aligned}$$

For (4.7), equation (3.4) implies

$$\begin{aligned} &\int_0^t (q_j + q_{j+1}) |\dot{X}_j - R_j|^2 + (q_j + q_{j-1}) |\dot{X}_{j-1} - R_{j-1}|^2 ds \\ &= 4 \int_0^t \mathcal{T}_j \cdot (R_j - R_{j-1}) - \frac{d}{dt} q_j \, ds \\ &\leq 4 \int_0^t \mathcal{T}_j \cdot \frac{2\pi}{L_h} \left(-\frac{\mathcal{N}_j q_j + \mathcal{N}_{j+1} q_{j+1}}{q_j + q_{j+1}} + \frac{\mathcal{N}_j q_j + \mathcal{N}_{j-1} q_{j-1}}{q_j + q_{j-1}} \right) ds + 4q_j(0) \\ &\leq C \int_0^t \left| \mathcal{T}_j \cdot \frac{\mathcal{N}_{j+1} q_{j+1}}{q_j + q_{j+1}} \right| + \left| \mathcal{T}_j \cdot \frac{\mathcal{N}_{j-1} q_{j-1}}{q_j + q_{j-1}} \right| ds + Ch \\ &\leq \varepsilon \int_0^t \frac{|\mathcal{T}_{j+1} - \mathcal{T}_j|^2}{q_j + q_{j+1}} + \frac{|\mathcal{T}_{j-1} - \mathcal{T}_j|^2}{q_j + q_{j-1}} ds + C(\varepsilon) \int_0^t \frac{q_{j+1}^2}{q_j + q_{j+1}} + \frac{q_{j-1}^2}{q_j + q_{j-1}} ds + Ch \\ &\leq \varepsilon \int_0^t \frac{|\mathcal{T}_{j+1} - \mathcal{T}_j|^2}{q_j + q_{j+1}} + \frac{|\mathcal{T}_{j-1} - \mathcal{T}_j|^2}{q_j + q_{j-1}} ds + C(\varepsilon)h \\ &= \frac{\varepsilon}{4} \int_0^t (q_j + q_{j+1}) |\dot{X}_j - R_j|^2 + (q_j + q_{j-1}) |\dot{X}_{j-1} - R_{j-1}|^2 ds + C(\varepsilon)h, \end{aligned}$$

where for the second inequality we used (4.3) and (2.1) to get that

$$\frac{2\pi}{L_h} \leq C, \quad q_j(0) = h_j |\partial_\xi X_h^0| = h_j |\partial_\xi I_h X^0| \leq C(X)h,$$

for the third inequality we employed Young's inequality and the fact $\mathcal{T}_j \cdot \mathcal{N}_j = 0$ to derive

$$\begin{aligned} \left| \mathcal{T}_j \cdot \frac{\mathcal{N}_{j+1} q_{j+1}}{q_j + q_{j+1}} \right| &= \left| (\mathcal{T}_j - \mathcal{T}_{j+1}) \cdot \frac{\mathcal{N}_{j+1} q_{j+1}}{q_j + q_{j+1}} \right| \\ &\leq |\mathcal{T}_j - \mathcal{T}_{j+1}| \frac{q_{j+1}}{q_j + q_{j+1}} \leq \varepsilon \frac{|\mathcal{T}_j - \mathcal{T}_{j+1}|^2}{q_j + q_{j+1}} + C(\varepsilon) \frac{q_{j+1}^2}{q_j + q_{j+1}}, \end{aligned}$$

and for the last equality we used (3.6). Obviously (4.7) follows by taking $\varepsilon = 1$. The estimates in (4.8) can be established by using arguments similar to those in [39, Lemma 4.1] and are omitted here for brevity. \square

Next we present the key length difference estimate with the aid of Lemma 4.3.

LEMMA 4.4. *Under the assumptions of Lemma 4.1, we have*

$$\|q - q_h\|_{L^2}^2 \leq C \int_0^t \int_{\mathbb{S}^1} |\partial_t X - \partial_t X_h|^2 q_h d\xi ds + C \int_0^t \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h d\xi ds + Ch^2,$$

where C is a constant depending on $C_p, C_P, \kappa_1, c_0, C_0, T^*$, and $K(X)$.

Proof. By definition, one has

$$\int_{\mathbb{S}^1} (q(\xi, t) - q_h(\xi, t))^2 d\xi = \sum_{j=1}^N \int_{I_j} (q(\xi, t) - q_j(t)/h_j)^2 d\xi.$$

By integration for $\frac{d}{dt} (h_j q - q_j)$, we can write

$$(h_j q - q_j)(t) = (h_j q - q_j)(0) + \int_0^t (h_j \partial_t q - \dot{q}_j) ds =: P + \int_0^t A ds,$$

where, by the interpolation error estimate (2.1) and inverse estimate (2.2), P satisfies

$$|P| = h_j \|\partial_\xi X^0\| - \|\partial_\xi I_h X^0\|_{I_j} \leq Ch \|\partial_\xi (X^0 - I_h X^0)\|_{L^\infty(I_j)} \leq Ch^{3/2} \|X^0\|_{H^2(I_j)}.$$

Applying (3.2) and (3.4), on each grid element $I_j = [\xi_{j-1}, \xi_j]$, we can write A as

$$\begin{aligned} A = \frac{d}{dt} (h_j q - q_j) &= - \left(\frac{h_j q}{2} |\partial_t X - R|^2 - \frac{q_j + q_{j+1}}{4} |\dot{X}_j - R_j|^2 \right) \\ &\quad - \left(\frac{h_j q}{2} |\partial_t X - R|^2 - \frac{q_j + q_{j-1}}{4} |\dot{X}_{j-1} - R_{j-1}|^2 \right) \\ &\quad - (h_j q (\partial_t X - R) \cdot R + \mathcal{T}_j \cdot (R_j - R_{j-1})) \triangleq: -A^+ - A^- - \widehat{A}. \end{aligned}$$

The terms $\int_0^t |A^+| ds$ and $\int_0^t |A^-| ds$ are estimated in [39, Lemma 4.4] and read as

$$\begin{aligned} \int_0^t (|A^+| + |A^-|) ds &\leq C Q_j + Ch \left(\int_0^t \sum_{k=j-1}^j |\partial_t X - R - (\dot{X}_k - R_k)|^2 ds \right)^{1/2}, \\ Q_j &:= \left(\int_0^t (|h_j q - q_j|^2 + |h_j q - q_{j+1}|^2 + |h_j q - q_{j-1}|^2) ds \right)^{1/2}, \end{aligned}$$

where (4.7) has been used. Applying (4.6), we immediately get

$$\begin{aligned} \int_0^t (|A^+| + |A^-|) ds &\leq C Q_j + Ch T_j + Ch Y_j + Ch \left(\int_0^t |L - L_h|^2 ds \right)^{1/2}, \\ T_j &:= \left(\int_0^t \sum_{k=j-1}^{j+1} |\mathcal{T} - \mathcal{T}_k|^2 ds \right)^{1/2}, \quad Y_j := \left(\int_0^t |\partial_t X - \dot{X}_j|^2 + |\partial_t X - \dot{X}_{j-1}|^2 ds \right)^{1/2}. \end{aligned}$$

It remains to estimate $\int_0^t |\widehat{A}| ds$. By definition, one has

$$\begin{aligned} \widehat{A} &= \frac{h_j q}{2} (\partial_t X - R) \cdot \left(-\frac{2\pi}{L} \mathcal{N} \right) - \frac{2\pi}{L_h} \frac{q_{j+1}}{q_j + q_{j+1}} \mathcal{T}_j \cdot \mathcal{N}_{j+1} \\ &\quad + \frac{h_j q}{2} (\partial_t X - R) \cdot \left(-\frac{2\pi}{L} \mathcal{N} \right) + \frac{2\pi}{L_h} \frac{q_{j-1}}{q_j + q_{j-1}} \mathcal{T}_j \cdot \mathcal{N}_{j-1} \triangleq: \widehat{A}_1 + \widehat{A}_2. \end{aligned}$$

Recalling (3.6), we observe that

$$\mathcal{T}_j \cdot \mathcal{N}_{j+1} = \mathcal{T}_j \cdot (\mathcal{T}_{j+1} - \mathcal{T}_j)^\perp = -\mathcal{N}_j \cdot (\mathcal{T}_{j+1} - \mathcal{T}_j) = -\frac{q_j + q_{j+1}}{2} \mathcal{N}_j \cdot (\dot{X}_j - R_j),$$

which implies

$$\begin{aligned}\widehat{A}_1 &= h_j q (\partial_t X - R) \cdot \left(-\frac{\pi}{L} \mathcal{N} + \frac{\pi}{L_h} \mathcal{N}_j \right) \\ &\quad + (q_{j+1} - h_j q) (\partial_t X - R) \cdot \frac{\pi}{L_h} \mathcal{N}_j + ((\dot{X}_j - R_j) - (\partial_t X - R)) \cdot \frac{\pi}{L_h} q_{j+1} \mathcal{N}_j.\end{aligned}$$

Therefore, by the assumptions and (4.6), we can estimate

$$\begin{aligned}\int_0^t |\widehat{A}_1| ds &\leq Ch \int_0^t \left| \frac{\pi \mathcal{N}}{L} - \frac{\pi \mathcal{N}_j}{L_h} \right| + |\dot{X}_j - R_j - \partial_t X + R| ds + C \int_0^t |q_{j+1} - h_j q| ds \\ &\leq Ch \int_0^t |L - L_h| + \sum_{k=j}^{j+1} |\mathcal{T} - \mathcal{T}_k| ds + C \int_0^t |q_{j+1} - h_j q| ds + Ch \int_0^t |\dot{X}_j - \partial_t X| ds \\ &\leq Ch \left(\int_0^t |L - L_h|^2 ds \right)^{1/2} + CQ_j + ChT_j + ChY_j,\end{aligned}$$

and similar estimates can be established for $\int_0^t |\widehat{A}_2| ds$.

To summarize, we obtain the following estimate on I_j :

$$|h_j q - q_j| \leq Ch^{3/2} \|X^0\|_{H^2(I_j)} + Ch \left(\int_0^t \|q - q_h\|_{L^2}^2 ds \right)^{1/2} + CQ_j + ChT_j + ChY_j,$$

where we have used (4.5). Applying (4.8), we get

$$\begin{aligned}\|h_j q(t) - q_j(t)\|_{L^2(I_j)}^2 &\leq Ch^4 \left(\|X^0\|_{H^2(I_j)}^2 + \int_0^t \|\partial_t X\|_{H^1(I_j)}^2 + \|X\|_{H^2(S_j)}^2 ds \right) \\ &\quad + Ch^2 \int_0^t h \|q - q_h\|_{L^2}^2 + \|q - q_h\|_{L^2(S_j)}^2 + \|\mathcal{T} - \mathcal{T}_h\|_{L^2(S_j)}^2 + \|\partial_t X - \partial_t X_h\|_{L^2(S_j)}^2 ds.\end{aligned}$$

Summing up over all grid elements I_j yields

$$Ch^2 \|q - q_h\|_{L^2}^2 \leq Ch^4 + Ch^2 \int_0^t \|q - q_h\|_{L^2}^2 + \|\mathcal{T} - \mathcal{T}_h\|_{L^2}^2 + \|\partial_t X - \partial_t X_h\|_{L^2}^2 ds,$$

where we have used the inequality

$$\|h_j q(t) - q_j(t)\|_{L^2(I_j)}^2 = h_j^2 \|q - q_h\|_{L^2(I_j)}^2 \geq Ch^2 \|q - q_h\|_{L^2(I_j)}^2.$$

Finally, the desired estimate is concluded by a Gronwall argument. \square

We are now in a position to prove Theorem 2.5.

Proof of Theorem 2.5. Since the nonlinear terms in (3.1) are locally Lipschitz with respect to X_j , the local existence and uniqueness are guaranteed by standard ODE theory. Let $T^* \in (0, T)$ be the maximal time such that the semidiscrete solution X_h exists and the following estimates hold:

$$(4.9) \quad \inf |\partial_\xi X_h| \geq \kappa_1/2 \quad \text{and} \quad \sup |\partial_\xi X_h| \leq 2\kappa_2 \quad \forall t \in [0, T^*].$$

Combining Lemmas 4.1 and 4.4 and employing Gronwall's argument, we can obtain that for any $t \in [0, T^*]$ the following holds:

$$(4.10) \quad \int_0^t \int_{S^1} |\partial_t X - \partial_t X_h|^2 q_h d\xi ds + \sup_{0 \leq s \leq t} \int_{S^1} |\mathcal{T} - \mathcal{T}_h|^2 q_h d\xi \leq Ch^2.$$

Plugging this back into Lemma 4.4, we obtain, for any $t \in [0, T^*]$,

$$(4.11) \quad \|q - q_h\|_{L^2}^2 \leq Ch^2.$$

By Lemma 4.2, there exists $h_0 > 0$ depending on $C_p, C_P, \kappa_1, \kappa_2, T$, and $K(X)$ such that for any $0 < h \leq h_0$, we have

$$\inf |\partial_\xi X_h| \geq 3\kappa_1/4 \quad \text{and} \quad \sup |\partial_\xi X_h| \leq 3\kappa_2/2 \quad \text{at } t = T^*.$$

By standard ODE theory, we can uniquely extend the above semidiscrete solution in a neighborhood of T^* . And by continuity, we obtain

$$\inf |\partial_\xi X_h| \geq \kappa_1/2 \quad \text{and} \quad \sup |\partial_\xi X_h| \leq 2\kappa_2 \quad \text{in a neighborhood of } T^*.$$

This contradicts the maximality of T^* , and thus $T^* = T$. Thus (4.9)–(4.11) hold for $t \in [0, T]$, and the nondegeneration property (2.6) follows by (4.9) and Assumption 2.1 by noticing that $q_j = h_j q_h$. We derive the error estimate by integration and (4.4):

$$\begin{aligned} \|X(\cdot, t) - X_h(\cdot, t)\|_{H^1}^2 &= \int_{\mathbb{S}^1} |X - X_h|^2 d\xi + \int_{\mathbb{S}^1} |\partial_\xi X - \partial_\xi X_h|^2 d\xi \\ &\leq 2 \int_{\mathbb{S}^1} \left(\int_0^t \partial_t X - \partial_t X_h ds \right)^2 d\xi + 2 \|X^0 - I_h X^0\|_{L^2}^2 + \|q - q_h\|_{L^2}^2 + \int_{\mathbb{S}^1} |\mathcal{T} - \mathcal{T}_h|^2 q q_h d\xi \\ &\leq 2 \int_{\mathbb{S}^1} T \int_0^t |\partial_t X - \partial_t X_h|^2 ds d\xi + Ch^2 \leq Ch^2, \end{aligned}$$

and the proof is completed. □

5. Numerical results. In this section, we present a fully discrete version of (2.4) to simulate the AP-CSF. Choose an integer m , and set the time step $\tau = T/m$, $t_k = k\tau$, $k = 0, \dots, m$. For simplicity, we choose a uniform mesh, i.e., $\xi_j = jh$ for $0 \leq j \leq N$ and $h = \frac{2\pi}{N}$. We take $X_h^0 = I_h X^0$. For $k \geq 1$, find $X_h^k = \sum_{j=1}^N X_j^k \varphi_j \in V_h$ by

$$\begin{aligned} &\int_{\mathbb{S}^1} |\partial_\xi X_h^{k-1}| \delta_\tau X_h^k \cdot v_h d\xi + \int_{\mathbb{S}^1} \partial_\xi X_h^k \cdot \partial_\xi v_h / |\partial_\xi X_h^{k-1}| d\xi \\ &+ \int_{\mathbb{S}^1} h^2 |\partial_\xi X_h^{k-1}| |\partial_\xi \delta_\tau X_h^k \cdot \partial_\xi v_h| / 6 d\xi + \int_{\mathbb{S}^1} 2\pi (\partial_\xi X_h^k)^\perp \cdot v_h / L_h^{k-1} d\xi = 0 \quad \forall v_h \in V_h, \end{aligned}$$

where δ_τ is the backward finite difference $\delta_\tau X_h^m = (X_h^m - X_h^{m-1})/\tau$, and L_h^{k-1} is the perimeter of the curve of X_h^{k-1} . Or, it can be written equivalently as a discretization for the ODE system (3.1):

$$\frac{q_j^{k-1} + q_{j+1}^{k-1}}{2\tau} (X_j^k - X_j^{k-1}) - \frac{X_{j+1}^k - X_j^k}{q_{j+1}^{k-1}} + \frac{X_j^k - X_{j-1}^k}{q_j^{k-1}} + \frac{\pi}{L_h^{k-1}} (X_{j+1}^k - X_{j-1}^k)^\perp = 0.$$

First, we test the convergence rates in the L^2 norm, H^1 seminorm, and the error of velocity, respectively. Since the exact solution of the AP-CSF (1.2) is unknown, we consider the following numerical errors:

$$\begin{aligned}
 (\mathcal{E}_1)_{h,\tau}(T) &:= \max_{1 \leq k \leq T/\tau} \|X_{h,\tau}^k - X_{h/2,\tau/4}^{4k}\|_{L^2(\mathbb{S}^1)}, \\
 (\mathcal{E}_2)_{h,\tau}(T) &:= \max_{1 \leq k \leq T/\tau} \|\partial_\xi X_{h,\tau}^k - \partial_\xi X_{h/2,\tau/4}^{4k}\|_{L^2(\mathbb{S}^1)}, \\
 (\mathcal{E}_3)_{h,\tau}(T) &:= \left(\sum_{k=0}^{T/\tau-1} \tau \left\| \frac{X_{h,\tau}^{k+1} - X_{h,\tau}^k}{\tau} - \frac{X_{h/2,\tau/4}^{4k+1} - X_{h/2,\tau/4}^{4k}}{\tau/4} \right\|_{L^2(\mathbb{S}^1)}^2 \right)^{1/2},
 \end{aligned}$$

where $X_{h,\tau}^k$ represents the solution obtained by the above fully discrete scheme with mesh size h and time step τ . The corresponding convergence order is defined as

$$\text{Order}_i = \log \left(\frac{(\mathcal{E}_i)_{h,\tau}(T)}{(\mathcal{E}_i)_{h/2,\tau/4}(T)} \right) / \log 2, \quad i = 1, 2, 3.$$

The errors and convergence orders are displayed in Table 5.1, where we choose $h = 2\pi/N$ and $\tau = 0.5h^2$ and the initial value is given by $X^0(\xi) = (2 \cos \xi, \sin \xi)$. The results indicate that the numerical solution converges linearly in space in the H^1 seminorm, which agrees with the theoretical analysis in Theorem 2.5. We can also observe that the solution and the velocity converge quadratically in $L_t^\infty L_x^2$ and $L_t^2 L_x^2$, respectively, which is superior to the analytical result in Theorem 2.5.

Finally, we check the structure-preserving properties of our algorithm, for which we take very tiny time step size τ such that the temporal error can be ignored. As is shown in Figure 5.1(a), the length of the curve is decreasing during the evolution, which confirms the theoretical analysis in Theorem 2.4. Furthermore, Figure 5.1(a) shows that the area is almost preserving, and, more specifically, Figure 5.1(b) indicates that the area enclosed by the curve has a numerical error at $\mathcal{O}(h^2)$. The evolution of the polygon with the number of grid points $N = 15$, which approximates the evolution of the ellipse determined by $x^2/4 + y^2 = 1$, is shown in Figure 5.2, from which we clearly see that the polygon remains convex during the evolution, which verifies the convexity-preserving property in Theorem 2.3.

TABLE 5.1
 Numerical errors up to $T = 1/4$.

N	$(\mathcal{E}_1)_{h,\tau}(1/4)$	Order ₁	$(\mathcal{E}_2)_{h,\tau}(1/4)$	Order ₂	$(\mathcal{E}_3)_{h,\tau}(1/4)$	Order ₃
16	2.08E-2	-	1.15E-0	-	3.09E-2	-
32	5.42E-3	1.94	6.01E-1	0.94	1.01E-2	1.61
64	1.37E-3	1.99	3.03E-1	0.99	2.76E-3	1.87
128	3.42E-4	2.00	1.52E-1	1.00	7.09E-4	1.96

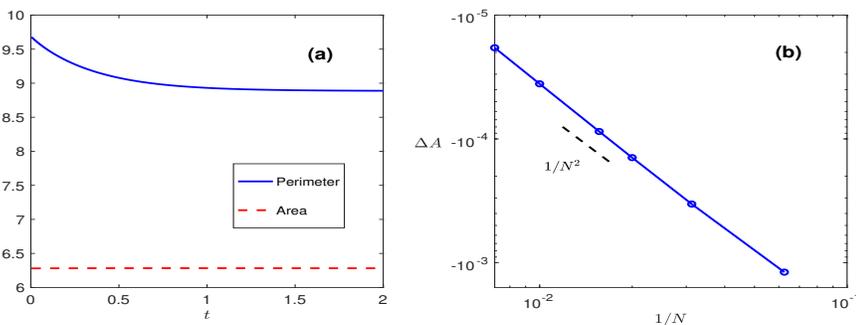


FIG. 5.1. Numerical results for an initial ellipse curve (i.e., $\frac{x^2}{4} + y^2 = 1$): (a) Evolution of the perimeter and area; (b) the area loss at $T = 1/4$ for different N .

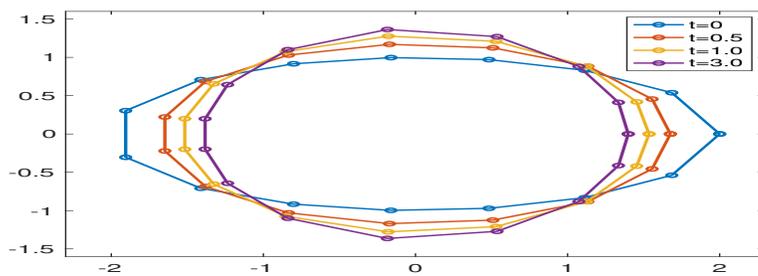


FIG. 5.2. The evolution of an initial convex polygon under the AP-CSF.

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