IMA Journal of Numerical Analysis (2025) **00**, 1–39
https://doi.org/10.1093/imanum/draf015

Convergence analysis of three semidiscrete numerical schemes for nonlocal geometric flows including perimeter terms

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[Received on 30 April 2024; revised on 9 September 2024]

We present and analyze three distinct semidiscrete schemes for solving nonlocal geometric flows incorporating perimeter terms. These schemes are based on the finite difference method, the finite element method and the finite element method with a specific tangential motion. We offer rigorous proofs of quadratic convergence under H^1-norm for the first scheme and linear convergence under H^1-norm for the latter two schemes. All error estimates rely on the observation that the error of the nonlocal term can be controlled by the error of the local term. Furthermore, we explore the relationship between the convergence under L^∞-norm and manifold distance. Extensive numerical experiments are conducted to verify the convergence analysis, and demonstrate the accuracy of our schemes under various norms for different types of nonlocal flows.

Keywords: nonlocal geometric flows; finite difference method; finite element method; tangential motion; error analysis; manifold distance.

1. Introduction

In this paper, we analyze and establish the convergence result of three distinct numerical methods for evolving a closed plane curve $\Gamma(t)$ under a nonlocal flow that involves perimeter. The normal velocity of $\Gamma(t)$ is determined by the formula

$$\mathscr{V} = (\kappa - f(L))\,\mathscr{N}, \tag{1.1}$$

where κ represents the curvature of the curve, f is a Lipschitz function, L is the perimeter, and $\mathscr{N}$ is the unit inner normal vector. Equation (1.1) encompasses a wide range of geometric flows, including:

$$f(L) = \begin{cases} \frac{2\pi}{L}, & \text{for area-preserving curve shortening flow of simple curves (Gage, 1986)},\\ \frac{2\pi\,\mathrm{ind}(\Gamma)}{L}, & \text{for area-preserving curve shortening flow of nonsimple curves (Wang \& Kong 2014)},\\ \frac{2\pi-\beta}{L}, & \text{for curve flows with a prescribed rate of change in the enclosed area (Dallaston \& McCue 2016; Tsai \& Wang 2018)}, \end{cases}$$

where $\beta \in (-\infty, \infty)$, $\mathrm{ind}(\Gamma) \in \mathbb{Z}$ denotes the rational index (Carmo, 2016) of a nonsimple curve Γ. The inclusion of an additional nonlocal force, $f(L)$, enables us to control the area change of an evolving curve. Indeed, by the theorem of turning tangents (do Carmo, 2016), the rate of area change can be determined by (Deckelnick *et al.*, 2005)

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \int_\Gamma \mathscr{V}\cdot\mathscr{N}\,\mathrm{d}s = \begin{cases} 0, & \text{for} \quad f(L) = \frac{2\pi}{L} \quad \text{and simple curves,} \\ 0, & \text{for} \quad f(L) = \frac{2\pi\,\mathrm{ind}(\Gamma)}{L} \quad \text{and nonsimple curves,} \\ -\beta, & \text{for} \quad f(L) = \frac{2\pi-\beta}{L} \quad \text{and simple curves.} \end{cases}$$

In this paper, we focus on the study of curve evolutions that maintain their topological characteristics. For simplicity, we always assume $L > 0$ during the whole evolution.

In recent years, significant attention has been paid to the theory development about nonlocal geometric flows. One prominent instance is the area-preserving curve shortening flow (AP-CSF), which is intricately connected to the mass-conserving Allen–Cahn equation (Rubinstein & Sternberg, 1992; Chen *et al.*, 2011), serving as its sharp-interface limit (Bronsard & Stoth, 1997). Additionally, AP-CSF is a key model for attachment-limited kinetics (Carter *et al.*, 1995; Dai *et al.*, 2010; Mugnai & Seis, 2013), which describes the growth dynamics of a solid phase surrounded by an undercooled liquid phase (Wagner, 1961). AP-CSF also plays a crucial role in image processing applications (Sapiro & Tannenbaum, 1995; Sapiro, 2001; Dolcetta *et al.*, 2002). Mathematically, the existence and convergence results for AP-CSF, applicable to both simple and nonsimple closed curves, have been extensively explored (Gage, 1986; Wang & Kong, 2014). Moreover, the study of curve flows with a prescribed rate of area change has emerged in the context of analyzing contracting bubbles in fluid mechanics and the Hele–Shaw problem (Dallaston & McCue, 2012, 2013, 2016), and its long-time evolution behavior was addressed in Tsai & Wang (2018).

Extensive numerical methods have been employed to simulate the AP-CSF and curve flows with a prescribed rate of change in the enclosed area. Examples of such methods for the AP-CSF include the finite difference method (FDM) (Mayer, 2000), the MBO method (Ruuth & Wetton, 2003), the crystalline algorithm (Ushijima & Yazaki, 2004), as well as PFEMs (Barrett *et al.*, 2020; Pei & Li, 2023). Additionally, a rescaled spectral collocation scheme was proposed in Dallaston & McCue (2016) for closed embedded plane curves with a prescribed rate of change in the enclosed area. However, there has been relatively little research on the numerical analysis of these methods. Recently, in Jiang *et al.* (2023), the authors proposed a semidiscrete finite element method (FEM) for the AP-CSF of simple curves and established its convergence in H^1-norm. In contrast to Dziuk's parametric FEM (Dziuk, 1994) for curve shortening flow (CSF), the nonlocal term in the geometric equations poses a major challenge for numerical analysis and computation. Specifically, the errors introduced by the nonlocal term, which involves the perimeter, are greatly influenced by the numerical errors resulting from the each length of polygonals, e.g., using the piecewise linear FEM to approximate the smooth curve. Therefore, it is crucial to carefully quantify the errors associated with length differences, as discussed in Sections 3 and 4. For recent advancements in parametric FEMs associated with CSF, we refer to Kovács *et al.* (2019); Li (2020); Ye & Cui (2021); Hu & Li (2022).

In this paper, we propose three numerical schemes for nonlocal geometric flows involving perimeter (1.1) and give their error estimates. Our main observation is that the difference between the nonlocal term and its discrete version can be managed through the disparity of the local term. Specifically, we introduce the following three different types of semidiscrete schemes:

- First, we employ FDM to discretize the parametrization equation of (1.1)

$$\partial_t X = \frac{1}{|\partial_\xi X|}\partial_\xi\left(\frac{\partial_\xi X}{|\partial_\xi X|}\right) - f(L)\left(\frac{\partial_\xi X}{|\partial_\xi X|}\right)^{\perp}, \quad \xi \in \mathbb{S}^1, \tag{1.2}$$

  where $\mathbb{S}^1 = [0, 2\pi]$, $(a, b)^{\perp} := (-b, a)$ denotes an anticlockwise rotation by $\pi/2$ and the periodic function $X(\xi, t) : \mathbb{S}^1 \times [0, T] \to \mathbb{R}^2$ is a parameterization of the closed curve $\Gamma \subset \mathbb{R}^2$. Under certain appropriate assumptions, we demonstrate that the resulting semidiscrete scheme converges quadratically in the discrete H^1-norm as defined in Deckelnick & Nürnberg (2023a). The proof is based on a careful Taylor expansion result and an averaged approximation of the normal vector.

- Secondly, we utilize an FEM for a natural weak formulation of (1.2). The derived semidiscrete scheme is based on our previous work on AP-CSF of simple curves (Jiang *et al.*, 2023). An H^1-optimal error estimate follows from our key observation mentioned above.

- Thirdly, we introduce an artificial tangential motion (TM) and apply an FEM for an alternative parametrization of the geometric equation

$$\partial_t X = \frac{\partial_{\xi\xi} X}{|\partial_\xi X|^2} - f(L)\mathcal{N}. \tag{1.3}$$

  This form of reparametrization was initially proposed by Deckelnick and Dziuk for the CSF (Deckelnick & Dziuk, 1995) to improve the mesh quality during evolution. It was later interpreted as a DeTurck trick by Elliot and Fritz in Elliott & Fritz (2017). Recently, the DeTurck trick has been further applied to various geometric flows such as elastic flow (Pozzi & Stinner, 2023), anisotropic CSF (Deckelnick & Nürnberg, 2023,b,c) and fourth-order flows (Deckelnick & Nürnberg, 2024). We emphasize that we have successfully extended the DeTurck trick to the general nonlocal flow case. The resulting semidiscrete scheme yields an asymptotic equidistribution property, as well as an H^1-optimal error estimate.

As a by-product, we further explore the convergence of the schemes under manifold distance, a topic extensively discussed in the numerical computation community (Bao & Zhao, 2021; Zhao *et al.*, 2021; Bao *et al.*, 2023; Jiang *et al.*, 2024a,b). We prove that, for simple curves, convergence in the function L^∞-norm implies convergence under the manifold distance. Moreover, we prove an optimal convergence of the finite difference scheme under the manifold distance.

The rest of this paper is organized as follows. In Section 2, we propose the semidiscrete schemes and provide the error estimates for the FDM. In Section 3, we consider the FEM, and the FEM-TM. Section 4 aims to establishing a connection between the convergence of the manifold distance and L^∞-norm. Section 5 presents extensive numerical experiments for the three different numerical schemes and various types of nonlocal flows. The numerical results demonstrate our convergence analysis results in both the H^1-norm and the manifold distance. Moreover, a better mesh quality is achieved for the finite element method with the aid of tangential motions. Finally, we draw some conclusions in Section 6.

We conclude this section with some comments on notations. Throughout the paper, the quantities related to the true solution are denoted as capital letters, while those related to the discrete solution are denoted as lowercase letters. Specifically, for the solution of (1.2), we denote $\mathscr{T} = \frac{\partial_\xi X}{|\partial_\xi X|}$ and $\mathscr{N} = \mathscr{T}^\perp$ by the unit tangent and inner normal of the curve, respectively. Therefore, (1.2) can be rewritten as

$$\partial_t X = \frac{1}{|\partial_\xi X|}\partial_\xi \mathscr{T} - f(L)\mathscr{N}, \quad \xi \in \mathbb{S}^1; \qquad X(\xi, 0) = X^0(\xi). \tag{1.4}$$

Throughout the paper, we maintain the orientation of parametrization X such that the rotation index $\mathrm{ind}(\Gamma)$ is a non-negative constant (Escher & Ito, 2005). For an embedded simple curve, this sign convention ensures that a unit circle has a positive constant curvature.

2. Finite difference method

In this section, we utilize an FDM to solve the equation (1.2). For spatial discretization, we utilize a uniform mesh, where the equidistributed grid points $\mathscr{G}_h := \{\xi_1, \ldots, \xi_N\} \subset \mathbb{S}^1$ are given by $\xi_j = jh, j = 0, \ldots, N$ for $h = 2\pi/N$ with $N \geq 2$. We use a periodic index, i.e., $a_j = a_{j\pm N}$ when involved. Denote $X_j = X(\xi_j)$, $\dot{X}_j = \partial_t X(\xi_j)$, and set

$$Q_j = |X_j - X_{j-1}|, \quad \mathscr{T}_j = \frac{X_j - X_{j-1}}{Q_j}, \quad j = 1, \ldots, N.$$

Let $x_h : \mathscr{G}_h \to \mathbb{R}^2$ be a grid function. We define the discrete length element q_j, the discrete tangent τ_j and normal n_j as

$$q_j = |x_j - x_{j-1}|, \quad \tau_j = \frac{x_j - x_{j-1}}{q_j}, \quad n_j = \tau_j^\perp, \tag{2.1}$$

where $x_j = x_h(\xi_j)$ denotes the vertex of the polygon that approximates the curve. Denote $l_h = \sum_{j=1}^{N} q_j$ by the perimeter of the polygon. Throughout the article, we denote C by a general constant which is independent of the mesh size h and might vary from line to line.

ASSUMPTION 2.1. Suppose that the solution of (1.2) satisfies $X \in C^1\left([0,T], C^4(\mathbb{S}^1)\right)$, i.e.,

$$K_1(X) := \|X\|_{C^1([0,T],C^4(\mathbb{S}^1))} < \infty,$$

and there exist constants $0 < C_1 < C_2$ such that

$$C_1 \leq \left|\partial_\xi X(\xi, t)\right| \leq C_2, \quad \forall\, (\xi, t) \in \mathbb{S}^1 \times [0, T]. \tag{2.2}$$

Under this assumption, we have the following results, which have been established in Deckelnick & Nürnberg (2023a).

LEMMA 1 (Deckelnick & Nürnberg, 2023a, Lemmas 3.1, 3.3). Under Assumption 2.1, there exists $h_0 > 0$ such that for $0 < h \le h_0$, the following expressions hold:

$$C_1 \le Q_j/h \le C_2, \quad \frac{Q_j + Q_{j+1}}{2h} = |\partial_\xi X(\xi_j)| + O(h^2), \tag{2.3a}$$

$$\mathscr{T}_j + \mathscr{T}_{j+1} = 2\,\mathcal{T}(\xi_j) + O(h^2), \quad \mathscr{T}_j = \frac{1}{2}\Big(\mathcal{T}(\xi_j) + \mathcal{T}(\xi_{j-1})\Big) + O(h^2), \tag{2.3b}$$

$$\frac{\mathscr{T}_{j+1} - \mathscr{T}_j}{h} = \partial_\xi \mathcal{T}(\xi_j) + O(h^2), \quad \frac{\dot{X}_{j+1} - \dot{X}_j}{h} = \frac{1}{2}\Big(\partial_t\partial_\xi X(\xi_j) + \partial_t\partial_\xi X(\xi_{j+1})\Big) + O(h^2), \tag{2.3c}$$

$$\Big|\tau_{j+1/2}^{\perp} - \mathcal{N}(\xi_j)\Big| \le \frac{2}{|\mathscr{T}_j + \mathscr{T}_{j+1}|}\Big(|\mathscr{T}_j - \tau_j| + |\mathscr{T}_{j+1} - \tau_{j+1}|\Big) + Ch^2, \tag{2.3d}$$

where $\tau_{j+1/2} := \frac{\tau_j + \tau_{j+1}}{|\tau_j + \tau_{j+1}|}$ represents the averaged vertex tangent.

For later use, the direct computation from (1.4) gives

$$\partial_t|\partial_\xi X| = \partial_t\partial_\xi X \cdot \mathcal{T} = \partial_\xi(\partial_t X \cdot \mathcal{T}) - \partial_t X \cdot \partial_\xi \mathcal{T} = -|\partial_\xi X||\partial_t X|^2 - f(L)|\partial_\xi X|\partial_t X \cdot \mathcal{N}. \tag{2.4}$$

For any grid function $u : \mathscr{G}_h \to \mathbb{R}^2$, or $Y \in C(\mathbb{S}^1)$, we define the backward difference quotient as

$$\delta u_j := \frac{u_j - u_{j-1}}{h}, \quad \delta Y_j = \frac{Y_j - Y_{j-1}}{h} = \frac{Y(\xi_j) - Y(\xi_{j-1})}{h}, \quad j = 1, \ldots, N.$$

Moreover, to measure the error, we introduce the following discrete norms:

$$\|u\|_{L^2_G} := \left(h\sum_{j=1}^{N}|u_j|^2\right)^{\frac{1}{2}}, \quad \|u\|_{H^1_G} := \left(h\sum_{j=1}^{N}\big(|u_j|^2 + |\delta u_j|^2\big)\right)^{\frac{1}{2}}. \tag{2.5}$$

DEFINITION 1. A semidiscrete finite difference approximation of (1.2) is to find a grid function $x_h : \mathscr{G}_h \times [0, T] \to \mathbb{R}^2$ such that

$$\dot{x}_j = \frac{2}{q_j + q_{j+1}}\Big(\tau_{j+1} - \tau_j\Big) - f(l_h)\tau_{j+1/2}^{\perp} \quad \text{in} \quad (0, T]; \qquad x_j(0) = X^0(\xi_j). \tag{2.6}$$

THEOREM 1. Let $X(\xi, t)$ be a solution of (1.2) that satisfies Assumption 2.1. Then, there exists a constant $h_0 > 0$ such that for all $0 < h \le h_0$, there is a unique finite difference semidiscrete solution $x_h(t)$ in the sense of (2.6). Furthermore, the following error estimate holds:

$$\sup_{t\in[0,T]} \|X(t) - x_h(t)\|_{H^1_G} \le Ch^2, \tag{2.7}$$

where the constants h_0 and C depend on C_1, C_2, $K_1(X)$, T and f.

Before giving the proof of Theorem 1, we compute the evolution equation for the discrete length q_j.

LEMMA 2. Suppose x_h is the finite difference semidiscrete solution in the sense of (2.6), then, it holds

$$\dot{q}_j + \tfrac{q_{j-1}+q_j}{4}|\dot{x}_{j-1}|^2 + \tfrac{q_j+q_{j+1}}{4}|\dot{x}_j|^2 + \tfrac{q_{j-1}+q_j}{4}f(l_h)\dot{x}_{j-1}\cdot\tau^\perp_{j-1/2} + \tfrac{q_{j+1}+q_j}{4}f(l_h)\dot{x}_j\cdot\tau^\perp_{j+1/2} = 0. \tag{2.8}$$

Proof. We begin by computing $\dot{q}_j$ as

$$\begin{aligned}
\dot{q}_j &= (\dot{x}_j - \dot{x}_{j-1})\cdot\tau_j \\
&= \tau_j\cdot\left(\frac{2}{q_j+q_{j+1}}(\tau_{j+1}-\tau_j) - f(l_h)\tau^\perp_{j+1/2} - \frac{2}{q_j+q_{j-1}}(\tau_j-\tau_{j-1}) + f(l_h)\tau^\perp_{j-1/2}\right) \\
&= \frac{2}{q_j+q_{j+1}}(\tau_j\cdot\tau_{j+1}-1) - f(l_h)\tau^\perp_{j+1/2}\cdot\tau_j - \frac{2}{q_j+q_{j-1}}(1-\tau_j\cdot\tau_{j-1}) + f(l_h)\tau^\perp_{j-1/2}\cdot\tau_j \\
&= \frac{2}{q_j+q_{j+1}}(\tau_j\cdot\tau_{j+1}-1) - f(l_h)\tau^\perp_{j+1/2}\cdot\tau_j + \frac{2}{q_j+q_{j-1}}(\tau_j\cdot\tau_{j-1}-1) - f(l_h)\tau^\perp_{j-1/2}\cdot\tau_{j-1} \\
&=: J_j + J_{j-1},
\end{aligned} \tag{2.9}$$

where for the last second equality, we have employed the property

$$\tau^\perp_{j-1/2}\cdot\tau_j = \tau^\perp_{j-1}/|\tau_j+\tau_{j-1}|\cdot\tau_j = -\tau_{j-1}\cdot\tau^\perp_j/|\tau_j+\tau_{j-1}| = -\tau_{j-1}\cdot\tau^\perp_{j-1/2}. \tag{2.10}$$

Multiplying (2.6) by $\frac{q_j+q_{j+1}}{4}f(l_h)\tau^\perp_{j+1/2}$, we obtain

$$\frac{q_j+q_{j+1}}{4}f(l_h)\tau^\perp_{j+1/2}\cdot\dot{x}_j + \frac{q_j+q_{j+1}}{4}f(l_h)^2 - f(l_h)\tau^\perp_{j+1/2}\cdot\frac{\tau_{j+1}-\tau_j}{2} = 0,$$

which can be simplified as

$$\frac{q_j+q_{j+1}}{4}f(l_h)\tau^\perp_{j+1/2}\cdot\dot{x}_j + \frac{q_j+q_{j+1}}{4}f(l_h)^2 + f(l_h)\tau^\perp_{j+1/2}\cdot\tau_j = 0, \tag{2.11}$$

by using (2.10). Combining (2.6) and (2.11), we get

$$\begin{aligned}
J_j &= \frac{2}{q_j+q_{j+1}}\left(-\frac{1}{2}|\tau_j-\tau_{j+1}|^2\right) - f(l_h)\tau^\perp_{j+1/2}\cdot\tau_j \\
&= -\frac{1}{q_j+q_{j+1}}\left|\dot{x}_j + f(l_h)\tau^\perp_{j+1/2}\right|^2\left(\frac{q_j+q_{j+1}}{2}\right)^2 - f(l_h)\tau^\perp_{j+1/2}\cdot\tau_j \\
&= -\frac{q_j+q_{j+1}}{4}\left(|\dot{x}_j|^2 + 2f(l_h)\tau^\perp_{j+1/2}\cdot\dot{x}_j + f(l_h)^2\right) - f(L_h)\tau^\perp_{j+1/2}\cdot\tau_j \\
&= -\frac{q_j+q_{j+1}}{4}|\dot{x}_j|^2 - \frac{q_j+q_{j+1}}{2}f(l_h)\tau^\perp_{j+1/2}\cdot\dot{x}_j - \frac{q_j+q_{j+1}}{4}f(l_h)^2 - f(l_h)\tau^\perp_{j+1/2}\cdot\tau_j \\
&= -\frac{q_j+q_{j+1}}{4}|\dot{x}_j|^2 - \frac{q_j+q_{j+1}}{4}f(l_h)\tau^\perp_{j+1/2}\cdot\dot{x}_j.
\end{aligned}$$

Plugging this into (2.9) yields (2.8), and the proof is completed. □

For the readers' convenience, we sketch the proof of Theorem 1. The main idea involves analyzing three types of errors: total error $\int_0^t h \sum_{j=1}^N |\dot{e}_j|^2 \, \mathrm{d}s$, the error in tangents $\sum_{j=1}^N q_j |\mathscr{T}_j - \tau_j|^2$ and the length difference error $\sum_{j=1}^N (Q_j - q_j)^2(t)$.

Outline of proof. The proof utilizes the continuity argument. Let $T^* > 0$ be the maximal time for which x_h solves (2.6) with a suitable estimate. For $t \in [0, T^*]$ we establish the following estimates:

(1) Let $\mathscr{R}_j$ and $\widetilde{\mathscr{R}}_j$ be the local truncation errors for (1.2) and (2.4), respectively. We show that

$$\mathscr{R}_j = O(h^2), \quad \widetilde{\mathscr{R}}_j = O(h^3).$$

(2) Denote $e_j(t) = X_j(t) - x_j(t)$. By subtracting (2.6) from (2.16), we derive the following stability estimate based on Step (1):

$$\int_0^t h \sum_{j=1}^N |\dot{e}_j|^2 \, \mathrm{d}s + \sup_{0 \le s \le t} \sum_{j=1}^N q_j |\mathscr{T}_j - \tau_j|^2 \le Ch^4 + C \int_0^t \frac{1}{h} \sum_{j=1}^N (Q_j - q_j)^2 \, \mathrm{d}s.$$

This step uses the key observation that the global perimeter difference can be reduced to a summation of local length differences, yielding:

$$|L - l_h| \le \sum_{j=1}^N |Q_j - q_j| + Ch^2.$$

(3) The length difference estimate implies

$$\frac{1}{h} \sum_{j=1}^N (Q_j - q_j)^2(t) \le C \int_0^t h \sum_{j=1}^N |\dot{e}_j|^2 \, \mathrm{d}s + C \sup_{0 \le s \le t} \sum_{j=1}^N q_j |\mathscr{T}_j - \tau_j|^2 + Ch^4.$$

Applying Step (2) again allows us to obtain the desired estimate (2.7) for $t \in (0, T^*]$.

Combining the above steps, the continuity argument enables us to extend the initial solution time T^* to T, ensuring that all the estimates hold in the interval $[0, T]$. □

Proof of Theorem 1. We define

$$T^* = \sup\left\{ t \in [0, T] : x_h \text{ solves (2.6) with } \tfrac{C_1}{2} \le \frac{q_j(t)}{h} \le 2C_2, \ \max_{j=1,\dots,N} |\mathscr{T}_j(t) - \tau_j(t)| \le h^{\frac{5}{4}} \right\}. \tag{2.12}$$

Clearly $T^* > 0$. Noticing the nonlinear terms in (2.6) are locally Lipschitz with respect to x_j, we get local existence and uniqueness using standard ODE theory. Furthermore, since $q_j(0) = Q_j(0)$ and $\tau_j(0) = \mathscr{T}_j(0)$, the desired estimate also holds by continuity. By (2.12) and the Lipschitz continuity of f, we have $\forall\, t \in [0, T^*]$,

$$2\pi C_1 \le L \le 2\pi C_2, \quad \pi C_1 \le l_h \le 4\pi C_2, \quad |f(L)| \le C, \tag{2.13}$$

where C is a constant, depending on C_1, C_2 and f. We claim that there exists a constant $h_1 > 0$ such that for $0 < h \le h_1$, it holds

$$\max_{j=1,\dots,N} |\dot{x}_j(t)| \le C, \quad \forall\, t \in [0, T^*], \tag{2.14}$$

where C depends on C_1, C_2 and f. Indeed, by (2.6), (2.3c), (2.12) and (2.13), we obtain

$$|\dot{x}_j|^2 \le 2\left|\frac{2}{q_j+q_{j+1}}\left(\tau_{j+1}-\tau_j\right)\right|^2 + 2\left|f(l_h)\tau^{\perp}_{j+1/2}\right|^2 \le C\left|\frac{\tau_{j+1}-\tau_j}{h}\right|^2 + C$$
$$\le C\left(\left|\frac{\mathscr{T}_{j+1}-\mathscr{T}_j}{h}\right| + \frac{2}{h}\max_{k=1,\dots,N}|\mathscr{T}_k-\tau_k|\right)^2 + C \le C.$$

Moreover, based on (2.3b), we have

$$\min_{j=1,\dots,N}|\mathscr{T}_j+\mathscr{T}_{j+1}| \ge 1, \tag{2.15}$$

when h is sufficiently small. Define the truncation error as

$$\mathscr{R}_j := \dot{X}_j - \frac{2}{Q_j+Q_{j+1}}\left(\mathscr{T}_{j+1}-\mathscr{T}_j\right) + f(L)\mathscr{N}(\xi_j), \tag{2.16}$$

$$\widetilde{\mathscr{R}}_j := \dot{Q}_j + \tfrac{Q_{j-1}+Q_j}{4}|\dot{X}_{j-1}|^2 + \tfrac{Q_j+Q_{j+1}}{4}|\dot{X}_j|^2 + \tfrac{Q_{j-1}+Q_j}{4}f(L)\dot{X}_{j-1}\cdot\mathscr{N}(\xi_{j-1}) + \tfrac{Q_{j+1}+Q_j}{4}f(L)\dot{X}_j\cdot\mathscr{N}(\xi_j). \tag{2.17}$$

(1) *Estimates of the truncation error* $\mathscr{R}_j, \widetilde{\mathscr{R}}_j$. Employing (1.2), (2.3a) and (2.3c), one gets

$$\begin{aligned}
\mathscr{R}_j &= \dot{X}_j - \frac{2}{Q_j+Q_{j+1}}\left(\mathscr{T}_{j+1}-\mathscr{T}_j\right) + f(L)\mathscr{N}(\xi_j)\\
&= \dot{X}_j - \frac{1}{|\partial_\xi X(\xi_j)| + O(h^2)}\left(\partial_\xi\mathscr{T}(\xi_j) + O(h^2)\right) + f(L)\mathscr{N}(\xi_j)\\
&= \dot{X}_j - \frac{1}{|\partial_\xi X(\xi_j)|}\left(1+O(h^2)\right)\cdot\left(\partial_\xi\mathscr{T}(\xi_j)+O(h^2)\right) + f(L)\mathscr{N}(\xi_j)\\
&= \dot{X}_j - \frac{1}{|\partial_\xi X(\xi_j)|}\partial_\xi\mathscr{T}(\xi_j) + f(L)\mathscr{N}(\xi_j) + O(h^2)\\
&= O(h^2).
\end{aligned} \tag{2.18}$$

Similarly, applying (2.4), (2.3b) and (2.3c) and using the regularity of X (see Assumption 2.1) again, we derive

$$\begin{aligned}
\dot{Q}_j &= \left(\dot{X}_j - \dot{X}_{j-1}\right)\cdot\mathscr{T}_j\\
&= \frac{h}{4}\left(\partial_t\partial_\xi X(\xi_{j-1}) + \partial_t\partial_\xi X(\xi_j)\right)\cdot\left(\mathscr{T}(\xi_j)+\mathscr{T}(\xi_{j-1})\right) + O(h^3)\\
&= \frac{h}{2}\partial_t\partial_\xi X(\xi_{j-1})\cdot\mathscr{T}(\xi_{j-1}) + \frac{h}{2}\partial_t\partial_\xi X(\xi_j)\cdot\mathscr{T}(\xi_j)\\
&\quad - \frac{h}{4}(\partial_t\partial_\xi X(\xi_j) - \partial_t\partial_\xi X(\xi_{j-1}))\cdot(\mathscr{T}(\xi_j)-\mathscr{T}(\xi_{j-1})) + O(h^3)\\
&= \frac{h}{2}\partial_t\partial_\xi X(\xi_{j-1})\cdot\mathscr{T}(\xi_{j-1}) + \frac{h}{2}\partial_t\partial_\xi X(\xi_j)\cdot\mathscr{T}(\xi_j) + O(h^3)\\
&= \frac{h}{2}\left(-|\partial_\xi X||\partial_t X|^2(\xi_{j-1}) - f(L)|\partial_\xi X|(\xi_{j-1})\partial_t X(\xi_{j-1})\cdot\mathscr{N}(\xi_{j-1})\right)\\
&\quad + \frac{h}{2}\left(-|\partial_\xi X||\partial_t X|^2(\xi_j) - f(L)|\partial_\xi X|(\xi_j)\partial_t X(\xi_j)\cdot\mathscr{N}(\xi_j)\right) + O(h^3),
\end{aligned}$$

which together with (2.3a) implies

$$\widetilde{\mathscr{R}}_j = \dot{Q}_j + \frac{Q_{j-1}+Q_j}{4}|\dot{X}_{j-1}|^2 + \frac{Q_j+Q_{j+1}}{4}|\dot{X}_j|^2 + \frac{Q_{j-1}+Q_j}{4} f(L)\dot{X}_{j-1}\cdot\mathscr{M}(\xi_{j-1}) + \frac{Q_{j+1}+Q_j}{4} f(L)\dot{X}_j\cdot\mathscr{M}(\xi_j) = O(h^3). \tag{2.19}$$

(2) *Stability*. Denote $e_j(t) = X_j(t) - x_j(t)$. Subtracting (2.6) from (2.16), one gets

$$\begin{aligned}
&\dot{e}_j - \frac{2}{q_j+q_{j+1}}\Big((\mathscr{T}_{j+1}-\tau_{j+1}) - (\mathscr{T}_j - \tau_j)\Big)\\
&= -f(L)\Big(\mathscr{M}(\xi_j) - \tau^{\perp}_{j+1/2}\Big) - \big(f(L) - f(l_h)\big)\,\tau^{\perp}_{j+1/2}\\
&\quad + 2\frac{(q_j - Q_j) + (q_{j+1} - Q_{j+1})}{(Q_j+Q_{j+1})(q_j+q_{j+1})}\Big(\mathscr{T}_{j+1} - \mathscr{T}_j\Big) + \mathscr{R}_j\\
&=: I^1_j + I^2_j + I^3_j + I^4_j.
\end{aligned}$$

Multiplying both sides with $\frac{1}{2}(q_j+q_{j+1})\dot{e}_j$ and summing together over all $j = 1,\dots,N$, we obtain

$$\frac{1}{2}\sum_{j=1}^{N}(q_j+q_{j+1})|\dot{e}_j|^2 - \sum_{j=1}^{N}\Big((\mathscr{T}_{j+1}-\tau_{j+1}) - (\mathscr{T}_j-\tau_j)\Big)\cdot\dot{e}_j = \sum_{k=1}^{4}\sum_{j=1}^{N}\frac{1}{2}(q_j+q_{j+1})I^k_j\cdot\dot{e}_j.$$

Applying (2.12), Young's inequality, Assumption 2.1 and (2.3a), we arrive at

$$\begin{aligned}
&-\sum_{j=1}^{N}\Big((\mathscr{T}_{j+1}-\tau_{j+1}) - (\mathscr{T}_j-\tau_j)\Big)\cdot\dot{e}_j\\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{N} q_j|\mathscr{T}_j - \tau_j|^2 + h\sum_{j=1}^{N}\left(\frac{Q_j-q_j}{Q_j}\,\delta\dot{X}_j\cdot(\mathscr{T}_j-\tau_j) + \frac{q_j}{2Q_j}\Big(\delta\dot{X}_j\cdot\mathscr{T}_j\Big)|\mathscr{T}_j-\tau_j|^2\right)\\
&\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{N} q_j|\mathscr{T}_j-\tau_j|^2 - C\sum_{j=1}^{N}\left(\frac{1}{h}(Q_j-q_j)^2 + q_j|\mathscr{T}_j-\tau_j|^2\right),
\end{aligned}$$

where for the first equality, we used the result in Deckelnick & Nürnberg (2023a) (cf. page 9 in Deckelnick & Nürnberg (2023a)). Employing (2.3d), (2.12), (2.13), (2.15) and Young's inequality, we get

$$\sum_{j=1}^{N}\frac{q_j+q_{j+1}}{2}I^1_j\cdot\dot{e}_j \leq Ch\sum_{j=1}^{N}\big|\mathscr{M}(\xi_j) - \tau^{\perp}_{j+1/2}\big||\dot{e}_j| \leq C(\varepsilon)h\sum_{j=1}^{N}|\mathscr{T}_j-\tau_j|^2 + \varepsilon h\sum_{j=1}^{N}|\dot{e}_j|^2 + C(\varepsilon)h^4.$$

Similarly, using (2.3a), (2.3c), Young's inequality and (2.18), one obtains

$$\sum_{j=1}^{N}\frac{1}{2}(q_j+q_{j+1})I_j^3\cdot\dot{e}_j\le C\sum_{j=1}^{N}\big(|Q_j-q_j|+|Q_{j+1}-q_{j+1}|\big)|\dot{e}_j|\le\varepsilon h\sum_{j=1}^{N}|\dot{e}_j|^2+\frac{C(\varepsilon)}{h}\sum_{j=1}^{N}|Q_j-q_j|^2,$$

$$\sum_{j=1}^{N}\frac{1}{2}(q_j+q_{j+1})I_j^4\cdot\dot{e}_j\le \varepsilon h\sum_{j=1}^{N}|\dot{e}_j|^2+C(\varepsilon)h^4.$$

It remains to estimate the term related to I_j^2. First, we estimate the error of the perimeter by applying the trapezoidal quadrature formula and (2.3a)

$$\begin{aligned}|L-l_h|&=\left|\int_{\mathbb{S}^1}|\partial_\xi X|\,\mathrm{d}\xi-\sum_{j=1}^{N}\frac{q_j+q_{j+1}}{2}\right|=\left|h\sum_{j=1}^{N}|\partial_\xi X|(\xi_j)+O(h^2)-\sum_{j=1}^{N}\frac{q_j+q_{j+1}}{2}\right|\\&=\left|\sum_{j=1}^{N}\frac{Q_j+Q_{j+1}}{2}+O(h^2)-\sum_{j=1}^{N}\frac{q_j+q_{j+1}}{2}\right|\le\sum_{j=1}^{N}|Q_j-q_j|+Ch^2.\end{aligned}\tag{2.20}$$

This immediately yields

$$\begin{aligned}\sum_{j=1}^{N}\frac{1}{2}(q_j+q_{j+1})I_j^2\cdot\dot{e}_j&\le Ch\sum_{j=1}^{N}|L-l_h||\dot{e}_j|\le C(\varepsilon)|L-l_h|^2+\varepsilon h\sum_{j=1}^{N}|\dot{e}_j|^2\\&\le C(\varepsilon)\left(\sum_{j=1}^{N}|Q_j-q_j|\right)^2+C(\varepsilon)h^4+\varepsilon h\sum_{j=1}^{N}|\dot{e}_j|^2\\&\le C(\varepsilon)\frac{1}{h}\sum_{j=1}^{N}|Q_j-q_j|^2+C(\varepsilon)h^4+\varepsilon h\sum_{j=1}^{N}|\dot{e}_j|^2.\end{aligned}$$

By combining the above inequalities, (2.12) and choosing ε to be sufficiently small, we are led to

$$h\sum_{j=1}^{N}|\dot{e}_j|^2+\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{N}q_j|\mathscr{T}_j-\tau_j|^2\le Ch^4+C\sum_{j=1}^{N}\left(\frac{1}{h}(Q_j-q_j)^2+q_j|\mathscr{T}_j-\tau_j|^2\right).$$

Through integration and utilizing Gronwall's inequality, we obtain

$$\int_0^t h\sum_{j=1}^{N}|\dot{e}_j|^2\,\mathrm{d}s+\sup_{0\le s\le t}\sum_{j=1}^{N}q_j|\mathscr{T}_j-\tau_j|^2\le Ch^4+C\int_0^t\frac{1}{h}\sum_{j=1}^{N}(Q_j-q_j)^2\,\mathrm{d}s,\tag{2.21}$$

for $0\le t\le T^*$, where C is a constant, depending on $C_1, C_2, K_1(X), T$ and f.

(3) *Length difference estimate*. By using (2.14) and (2.20), we can derive the following estimates

$$\left(|\dot{x}_j|^2 - |\dot{X}_j|^2\right) \le (|\dot{x}_j| + |\dot{X}_j|)|\dot{x}_j - \dot{X}_j| \le C|\dot{e}_j|, \quad f(l_h) - f(L) \le C\sum_{j=1}^{N}|Q_j - q_j| + Ch^2.$$

Subtracting (2.17) from (2.8), integrating from 0 to t, and applying (2.3d) together with the above estimate, we get

$$\begin{aligned}|Q_j - q_j|(t) &\le \int_0^t |\dot{Q}_j - \dot{q}_j|(s)\,\mathrm{d}s + |Q_j - q_j|(0)\\ &\le C\int_0^t |q_j - Q_j| + |q_{j+1} - Q_{j+1}| + |q_{j-1} - Q_{j-1}|\,\mathrm{d}s\\ &\quad + Ch\int_0^t |\tau_j - \mathscr{T}_j| + |\tau_{j+1} - \mathscr{T}_{j+1}| + |\tau_{j-1} - \mathscr{T}_{j-1}|\,\mathrm{d}s\\ &\quad + Ch\int_0^t \sum_{j=1}^{N}|Q_j - q_j|\,\mathrm{d}s + Ch^3 + Ch\int_0^t |\dot{e}_{j-1}| + |\dot{e}_j|\,\mathrm{d}s + \int_0^t |\widetilde{\mathscr{R}}_j|\,\mathrm{d}s.\end{aligned}$$

This together with (2.19) yields

$$\frac{1}{h}\sum_{j=1}^{N}(Q_j - q_j)^2(t) \le C\int_0^t h\sum_{j=1}^{N}|\dot{e}_j|^2\,\mathrm{d}s + C\int_0^t \frac{1}{h}\sum_{j=1}^{N}(Q_j - q_j)^2\,\mathrm{d}s + C\int_0^t h\sum_{j=1}^{N}|\mathscr{T}_j - \tau_j|^2\,\mathrm{d}s + Ch^4.$$

Applying Gronwall's inequality, we get

$$\begin{aligned}\frac{1}{h}\sum_{j=1}^{N}(Q_j - q_j)^2(t) &\le C\int_0^t h\sum_{j=1}^{N}|\dot{e}_j|^2\,\mathrm{d}s + C\int_0^t \sum_{j=1}^{N} q_j|\mathscr{T}_j - \tau_j|^2\,\mathrm{d}s + Ch^4\\ &\le C\int_0^t h\sum_{j=1}^{N}|\dot{e}_j|^2\,\mathrm{d}s + C\sup_{0\le s\le t}\sum_{j=1}^{N} q_j|\mathscr{T}_j - \tau_j|^2 + Ch^4\\ &\le Ch^4 + C\int_0^t \frac{1}{h}\sum_{j=1}^{N}(Q_j - q_j)^2(s)\,\mathrm{d}s, \end{aligned} \tag{2.22}$$

where for the last inequality we utilized (2.21). Hence Gronwall's inequality gives

$$\frac{1}{h}\sum_{j=1}^{N}(Q_j - q_j)^2(t) \le Ch^4, \quad 0 \le t \le T^*. \tag{2.23}$$

This together with (2.21) implies

$$\int_0^{T^*} h\sum_{j=1}^{N}|\dot{e}_j|^2\,\mathrm{d}s+\sup_{0\le t\le T^*}\sum_{j=1}^{N}q_j|\mathscr{T}_j-\tau_j|^2\le Ch^4. \tag{2.24}$$

Now we are ready to complete the proof by a continuity argument. It follows from (2.24) that there exists $h_2>0$ such that when $h\le h_2$,

$$|\mathscr{T}_j-\tau_j|(t)\le h^{-\frac{1}{2}}\left(h\sum_{k=1}^{N}|\mathscr{T}_k-\tau_k|^2(t)\right)^{\frac{1}{2}}\le Ch^{-\frac{1}{2}}h^2\le \tfrac{1}{2}h^{\frac{5}{4}},\quad 0\le t\le T^*.$$

On the other hand, it can be easily derived from (2.23) that

$$|Q_j(t)-q_j(t)|\le Ch^{3/2},\quad 0\le t\le T^*,$$

which together with (2.3a) yields

$$\tfrac{2}{3}C_1\le q_j(t)/h\le \tfrac{3}{2}C_2,\quad h\le h_3.$$

By continuity we can extend T^* such that

$$\tfrac{C_1}{2}\le\frac{q_j(t)}{h}\le 2C_2,\quad \max_{j=1,\ldots,N}|\mathscr{T}_j(t)-\tau_j(t)|\le h^{\frac{5}{4}}.$$

This contradicts (2.12) if $T^*<T$. Therefore, $T^*=T$. As for the estimate of e_j, we first notice

$$\delta e_j=\delta X_j-\delta x_j=\frac{Q_j(\mathscr{T}_j-\tau_j)}{h}+\frac{(Q_j-q_j)}{h}\tau_j.$$

Recalling (2.23) and (2.24), we immediately get

$$h\sum_{j=1}^{N}|e_j|^2\le C\int_0^t h\sum_{j=1}^{N}|\dot{e}_j|^2\,\mathrm{d}s\le Ch^4,\quad h\sum_{j=1}^{N}|\delta e_j|^2\le Ch\sum_{j=1}^{N}\left(|\mathscr{T}_j-\tau_j|^2+\frac{(Q_j-q_j)^2}{h^2}\right)\le Ch^4,$$

which yields

$$\|X(t)-x_h(t)\|_{H^1_G}=\left(h\sum_{j=1}^{N}\big(|e_j|^2+|\delta e_j|^2\big)\right)^{\frac{1}{2}}\le Ch^2,\quad 0\le t\le T,$$

and the proof is completed by taking $h_0=\min\{h_1,h_2,h_3\}$. □

3. Finite element methods

In this section, we present two FEMs based on different formulations and establish their error estimates. The parametrization (1.2) naturally leads to a weak formulation: for any $v \in (H^1(\mathbb{S}^1))^2$, it holds

$$\int_{\mathbb{S}^1} |\partial_\xi X| \partial_t X \cdot v \, \mathrm{d}\xi + \int_{\mathbb{S}^1} \mathscr{T} \cdot \partial_\xi v \, \mathrm{d}\xi + \int_{\mathbb{S}^1} f(L)(\partial_\xi X)^\perp \cdot v \, \mathrm{d}\xi = 0. \tag{3.1}$$

For spatial discretization, let $0 = \xi_0 < \xi_1 < \ldots < \xi_N = 2\pi$ be a partition of $\mathbb{S}^1$. We denote $h_j = \xi_j - \xi_{j-1}$ as the length of the interval $I_j := [\xi_{j-1}, \xi_j]$ and $h = \max_j h_j$. We assume that the partition and the exact solution are regular in the following senses, respectively:

Assumption 3.1. There exist constants c_p and c_P such that

$$\min_j h_j \geq c_p h, \quad |h_{j+1} - h_j| \leq c_P h^2, \quad 1 \leq j \leq N.$$

Assumption 3.2. Suppose the solution of (1.2) satisfies $X \in W^{1,\infty}\big([0,T], H^2(\mathbb{S}^1)\big)$, i.e.,

$$K_2(X) := \|X\|_{W^{1,\infty}([0,T],H^2(\mathbb{S}^1))} < \infty,$$

and there exist constants $0 < C_1 < C_2$ such that (2.2) holds.

We define the following finite element space consisting of piecewise linear functions satisfying periodic boundary conditions:

$$V_h = \left\{ v \in C(\mathbb{S}^1, \mathbb{R}^2) : v|_{I_j} \in P_1(I_j), \quad 1 \leq j \leq N, \quad v(\xi_0) = v(\xi_N) \right\},$$

where P_1 denotes all polynomials with degrees at most 1. For any continuous function $v \in C(\mathbb{S}^1, \mathbb{R}^2)$, the linear interpolation $I_h v \in V_h$ is uniquely determined through $I_h v(\xi_j) = v(\xi_j)$ for all $1 \leq j \leq N$ and can be explicitly written as $I_h v(\xi) = \sum_{j=1}^{N} v(\xi_j)\varphi_j(\xi)$, where φ_j represents the standard Lagrange basis function satisfying $\varphi_j(\xi_i) = \delta_{ij}$.

3.1 *FEM with only the normal motion*

In this part, we present an FEM based on the original parametrization (1.2).

Definition 2. We call a function

$$x_h(\xi, t) = \sum_{j=1}^{N} x_j(t)\varphi_j(\xi) : \mathbb{S}^1 \times [0,T] \to \mathbb{R}^2 \tag{3.2}$$

is a semidiscrete solution of (1.2) if it satisfies $x_h(\xi,0) = I_h X^0$ and for all $v_h \in V_h$, it holds

$$\int_{\mathbb{S}^1} q_h \partial_t x_h \cdot v_h \,\mathrm{d}\xi + \int_{\mathbb{S}^1} \tau_h \cdot \partial_\xi v_h \,\mathrm{d}\xi + \int_{\mathbb{S}^1} \frac{\mathbf{h}^2 q_h}{6} \partial_\xi \partial_t x_h \cdot \partial_\xi v_h \,\mathrm{d}\xi + \int_{\mathbb{S}^1} f(l_h)(\partial_\xi x_h)^\perp \cdot v_h \,\mathrm{d}\xi = 0, \tag{3.3}$$

where

$$q_h = |\partial_\xi x_h| = \sum_{j=1}^N \frac{q_j}{h_j}\chi_{I_j}, \quad \tau_h = \frac{\partial_\xi x_h}{|\partial_\xi x_h|} = \sum_{j=1}^N \frac{x_j - x_{j-1}}{q_j}\chi_{I_j}, \tag{3.4}$$

represent the discrete length element and unit tangent vector, respectively, l_h represents the perimeter of the evolved polygon with vertices x_j, and $\mathbf{h} = \sum_{j=1}^N h_j \chi_{I_j}$ with χ being the characteristic function.

Remark 1. Compared to the original formulation (3.1), here an extra term $\int_{\mathbb{S}^1} \frac{\mathbf{h}^2 |\partial_\xi x_h|}{6} \partial_\xi \partial_t x_h \cdot \partial_\xi v_h \,\mathrm{d}\xi$ is introduced in (3.3), which reduces to the so-called mass-lumped scheme (3.5). Clearly, this term does not affect the convergence order for a linear FEM. As was interpreted in Dziuk (1999); Jiang *et al.* (2023), this mass-lumped version can preserve the length shortening property for the CSF/AP-CSF, which was missing for the original formula.

Taking $v_h = (\varphi_j, 0)$ and $v_h = (0, \varphi_j)$ for $j = 1, \ldots, N$ in (3.3), we are led to the following $2N$ ordinary differential equations:

$$\frac{q_j + q_{j+1}}{2}\dot{x}_j = \tau_{j+1} - \tau_j - f(l_h)(x_{j+1} - x_{j-1})^\perp, \tag{3.5}$$

where τ_j is the discrete tangent defined as (2.1). Furthermore, we have the following identities

$$\dot{q}_j = -\frac{1}{q_j + q_{j+1}}|\tau_{j+1} - \tau_j|^2 - \frac{1}{q_j + q_{j-1}}|\tau_{j-1} - \tau_j|^2 + \tau_j \cdot \left(r_j - r_{j-1}\right) \tag{3.6}$$

$$= -\frac{q_j + q_{j+1}}{4}|\dot{x}_j - r_j|^2 - \frac{q_j + q_{j-1}}{4}|\dot{x}_{j-1} - r_{j-1}|^2 + \tau_j \cdot \left(r_j - r_{j-1}\right), \tag{3.7}$$

where for simplicity we denote

$$r_j = -f(l_h)\frac{n_j q_j + n_{j+1} q_{j+1}}{q_j + q_{j+1}}. \tag{3.8}$$

Theorem 2. Let $X(\xi, t)$ be a solution of (1.2) satisfying Assumption 3.2. Assume that the partition of $\mathbb{S}^1$ satisfies Assumption 3.1. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$, there exists a unique semidiscrete solution x_h for (3.3). Furthermore, the solution satisfies

$$\int_0^T \|\partial_t X - \partial_t x_h\|_{L^2}^2 \,\mathrm{d}t + \sup_{t\in[0,T]} \|X - x_h\|_{H^1}^2 \le Ch^2, \tag{3.9}$$

where h_0 and C depend on $c_p, c_P, C_1, C_2, T, K_2(X)$ and f.

Before presenting the proof of Theorem 2, we first list a lemma which will be used later.

LEMMA 3 (Jiang *et al.*, 2023, Lemma 4.2). Under Assumptions 3.1 and 3.2, suppose further

$$\int_{\mathbb{S}^1} \left|\mathscr{T} - \tau_h\right|^2 q_h \,\mathrm{d}\xi + \| |\partial_\xi X| - q_h \|_{L^2}^2 \le Ch^2, \quad \forall\, t \in [0, T^*],$$

then there exists a constant h_0 such that for any $0 < h \le h_0$, it holds

$$\inf_\xi q_h \ge 3C_1/4, \quad \sup_\xi q_h \le 3C_2/2, \quad \forall\, t \in [0, T^*],$$

where C_1 and C_2 are the lower and upper bounds of $|\partial_\xi X|$ shown in (2.2).

Proof of Theorem 2. Similar to the proof of Theorem 1, we apply the continuity argument. Define

$$T^* = \sup\{t \in [0, T] : (3.3) \text{ has a unique solution } x_h \text{ and } \inf q_h \ge C_1/2,\ \sup q_h \le 2C_2\}. \tag{3.10}$$

Since the nonlinear terms in (3.5) are locally Lipschitz with respect to x_j, the local existence and uniqueness follow from standard ODE theory, and thus $T^* > 0$. Moreover, due to the Lipschitz property of f and Assumption (3.10), for any $t \in [0, T^*]$, it holds that

$$2\pi C_1 \le L \le 2\pi C_2, \quad \pi C_1 \le l_h \le 4\pi C_2, \quad |f(L)| \le C, \tag{3.11}$$

where C is a constant, depending on C_1, C_2, f.

(1) *Stability*. Taking the difference between (3.1) and (3.3), and choosing $v_h = I_h(\partial_t X) - \partial_t x_h \in V_h$, we get

$$\begin{aligned}
&\int_{\mathbb{S}^1} |\partial_t X - \partial_t x_h|^2 q_h \,\mathrm{d}\xi + \int_{\mathbb{S}^1} \left(\mathscr{T} - \tau_h\right) \left(\partial_\xi \partial_t X - \partial_\xi \partial_t x_h\right) \mathrm{d}\xi \\
&= \int_{\mathbb{S}^1} \partial_t X \cdot (q_h - |\partial_\xi X|) \left(I_h \partial_t X - \partial_t x_h\right) \mathrm{d}\xi + \int_{\mathbb{S}^1} \frac{\mathbf{h}^2 q_h}{6} \partial_\xi \partial_t x_h \cdot \partial_\xi \left(I_h \partial_t X - \partial_t x_h\right) \mathrm{d}\xi \\
&\quad + \int_{\mathbb{S}^1} q_h (\partial_t X - \partial_t x_h) \cdot (\partial_t X - I_h \partial_t X) \,\mathrm{d}\xi + \int_{\mathbb{S}^1} \left(\mathscr{T} - \tau_h\right) \cdot \left(\partial_\xi \partial_t X - \partial_\xi I_h \partial_t X\right) \mathrm{d}\xi \\
&\quad + \int_{\mathbb{S}^1} f(L) \left(\partial_\xi X - \partial_\xi x_h\right)^\perp \cdot \left(\partial_t x_h - I_h \partial_t X\right) \mathrm{d}\xi \\
&\quad + \int_{\mathbb{S}^1} \left(f(L) - f(l_h)\right) \left(\partial_\xi x_h\right)^\perp \cdot \left(\partial_t x_h - I_h \partial_t X\right) \mathrm{d}\xi =: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned}$$

The estimates of the second term on the left side and J_j for $1 \le j \le 4$ can be found in Dziuk (1999, Lemma 5.1) or Jiang *et al.* (2023, Lemma 4.1), which can be summarized as follows:

$$\begin{aligned}
&\int_{\mathbb{S}^1} (\mathscr{T} - \tau_h) \cdot \left(\partial_\xi \partial_t X - \partial_\xi \partial_t x_h\right) \mathrm{d}\xi \\
&\quad \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{S}^1} |\mathscr{T} - \tau_h|^2 q_h \,\mathrm{d}\xi\right) - C\|\partial_\xi \partial_t X\|_{L^\infty}\left(\int_{\mathbb{S}^1} |\mathscr{T} - \tau_h|^2 q_h \,\mathrm{d}\xi + \||\partial_\xi X| - q_h\|_{L^2}^2\right), \\
&J_1 + J_2 + J_3 + J_4 \le \varepsilon \int_{\mathbb{S}^1} \left|\partial_t X - \partial_t X_h\right|^2 q_h \,\mathrm{d}\xi + C(\varepsilon)\left\|\partial_t X\right\|_{L^\infty}^2 \||\partial_\xi X| - q_h\|_{L^2}^2 + C(\varepsilon) h^2 \|\partial_t X\|_{H^1}^2 \\
&\qquad\qquad + C\int_{\mathbb{S}^1} |\mathscr{T} - \tau_h|^2 q_h \,\mathrm{d}\xi,
\end{aligned}$$

where ε is a generic positive constant which will be chosen later. For J_5 and J_6, in view of the Lipschitz property of f, (3.11), and the identity

$$|\partial_\xi X - \partial_\xi x_h|^2 = (|\partial_\xi X| - q_h)^2 + |\partial_\xi X| q_h |\mathscr{T} - \tau_h|^2, \tag{3.12}$$

applying similar techniques in Jiang *et al.* (2023) (cf. proof of Lemma 4.1), we can get

$$\begin{aligned}
J_5 &= \int_{\mathbb{S}^1} f(L)\left(\partial_\xi X - \partial_\xi x_h\right)^\perp \cdot \left(\partial_t X - I_h \partial_t X\right) \mathrm{d}\xi + \int_{\mathbb{S}^1} f(L)\left(\partial_\xi X - \partial_\xi x_h\right)^\perp \cdot \left(\partial_t x_h - \partial_t X\right) \mathrm{d}\xi \\
&\le C\left\|\partial_\xi X\right\|_{L^\infty} \int_{\mathbb{S}^1} \left|\mathscr{T} - \tau_h\right|^2 q_h \,\mathrm{d}\xi + C\int_{\mathbb{S}^1} (|\partial_\xi X| - q_h)^2 \,\mathrm{d}\xi + Ch^2 \|\partial_t X\|_{H^1}^2 \\
&\quad + C(\varepsilon)\left\|\partial_\xi X\right\|_{L^\infty} \int_{\mathbb{S}^1} \left|\mathscr{T} - \tau_h\right|^2 q_h \,\mathrm{d}\xi + C(\varepsilon)\||\partial_\xi X| - q_h\|_{L^2}^2 + \varepsilon \int_{\mathbb{S}^1} \left|\partial_t x_h - \partial_t X\right|^2 q_h \,\mathrm{d}\xi, \\
J_6 &= \int_{\mathbb{S}^1} \left(f(L) - f(l_h)\right)\left(\partial_\xi x_h\right)^\perp \cdot \left(\partial_t X - I_h \partial_t X\right) \mathrm{d}\xi + \int_{\mathbb{S}^1} \left(f(L) - f(l_h)\right)\left(\partial_\xi x_h\right)^\perp \cdot \left(\partial_t x_h - \partial_t X\right) \mathrm{d}\xi \\
&\le C|L - l_h|^2 + C\|\partial_t X - I_h \partial_t X\|_{L^2}^2 + C(\varepsilon)|L - l_h|^2 + \varepsilon \int_{\mathbb{S}^1} q_h \left|\partial_t x_h - \partial_t X\right|^2 \mathrm{d}\xi \\
&\le C(\varepsilon)\||\partial_\xi X| - q_h\|_{L^2}^2 + Ch^2 \|\partial_t X\|_{H^1}^2 + \varepsilon \int_{\mathbb{S}^1} q_h \left|\partial_t x_h - \partial_t X\right|^2 \mathrm{d}\xi.
\end{aligned}$$

Here we use the inequalities $|L - l_h|^2 \le \||\partial_\xi X| - q_h\|_{L^1}^2 \le C\||\partial_\xi X| - q_h\|_{L^2}^2$. Combining all the above estimates, we are led to

$$\begin{aligned}
&\int_{\mathbb{S}^1} |\partial_t X - \partial_t x_h|^2 q_h \,\mathrm{d}\xi + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^1} |\mathscr{T} - \tau_h|^2 q_h \,\mathrm{d}\xi \le 4\varepsilon \int_{\mathbb{S}^1} |\partial_t X - \partial_t x_h|^2 q_h \,\mathrm{d}\xi \\
&\qquad + C(\varepsilon) h^2 \|\partial_t X\|_{H^2}^2 + C(\varepsilon, K_2(X))\||\partial_\xi X| - q_h\|_{L^2}^2 + C(\varepsilon, K_2(X))\int_{\mathbb{S}^1} |\mathscr{T} - \tau_h|^2 q_h \,\mathrm{d}\xi.
\end{aligned}$$

Choosing ε small enough, integrating both sides with respect to time from 0 to t and applying Gronwall's argument, we arrive at

$$\int_0^t \int_{\mathbb{S}^1} |\partial_t X - \partial_t x_h|^2 q_h \,\mathrm{d}\xi \,\mathrm{d}s + \sup_{0\le s\le t} \int_{\mathbb{S}^1} |\mathscr{T} - \tau_h|^2 q_h \,\mathrm{d}\xi \le C \int_0^t \| |\partial_\xi X| - q_h \|_{L^2}^2 \,\mathrm{d}s + Ch^2, \tag{3.13}$$

where C is a constant, depending on c_p, c_P, C_1, C_2, T, $K_2(X)$ and f.

(2) *Length difference estimate*. Applying the Lipschitz property of f and (3.11), a mild modification of the proof of Jiang *et al.* (2023, Lemma 4.3, Lemma 4.4) enables us to establish the same length difference estimate as in Jiang *et al.* (2023):

$$\| |\partial_\xi X| - q_h \|_{L^2}^2 \le C \int_0^t \int_{\mathbb{S}^1} \left|\partial_t X - \partial_t x_h\right|^2 q_h \,\mathrm{d}\xi \,\mathrm{d}s + C \int_0^t \int_{\mathbb{S}^1} \left|\mathscr{T} - \tau_h\right|^2 q_h \,\mathrm{d}\xi \,\mathrm{d}s + Ch^2, \tag{3.14}$$

where C depends on c_p, c_P, C_1, C_2, T, $K_2(X)$ and f. For the details, we refer to Jiang *et al.* (2023).

Combining (3.13) and (3.14), employing Gronwall's inequality, we derive

$$\int_0^t \int_{\mathbb{S}^1} \left|\partial_t X - \partial_t x_h\right|^2 q_h \,\mathrm{d}\xi \,\mathrm{d}s + \sup_{0\le s\le t} \int_{\mathbb{S}^1} \left|\mathscr{T} - \tau_h\right|^2 q_h \,\mathrm{d}\xi \le Ch^2, \quad \forall\, t \in [0, T^*], \tag{3.15}$$

which together with (3.14) yields

$$\| |\partial_\xi X| - q_h \|_{L^2}^2 \le Ch^2.$$

Applying Lemma 3, there exists $h_0 > 0$, depending on $c_p, c_P, C_1, C_2, T, K_2(X)$ such that for any $0 < h \le h_0$, we have

$$\inf q_h \ge 3C_1/4 \quad \text{and} \quad \sup q_h \le 3C_2/2, \quad t \in [0, T^*].$$

By standard ODE theory, we can uniquely extend the above semidiscrete solution in a neighborhood of T^*, and thus $T^* = T$. The estimate (3.9) can be concluded similarly as in (Jiang *et al.*, 2023, Theorem 2.5) by integration, (3.12) and (3.15):

$$\begin{aligned}
\|X(\cdot,t) - x_h(\cdot,t)\|_{H^1}^2 &= \int_{\mathbb{S}^1} |X - x_h|^2 \,\mathrm{d}\xi + \int_{\mathbb{S}^1} |\partial_\xi X - \partial_\xi x_h|^2 \,\mathrm{d}\xi \\
&\le 2\int_{\mathbb{S}^1} \left(\int_0^t \partial_t X - \partial_t x_h \,\mathrm{d}s\right)^2 \mathrm{d}\xi + 2\|X^0 - I_h X^0\|_{L^2}^2 + \| |\partial_\xi X| - q_h \|_{L^2}^2 + \int_{\mathbb{S}^1} |\mathscr{T} - \tau_h|^2 |\partial_\xi X| q_h \,\mathrm{d}\xi \\
&\le 2\int_{\mathbb{S}^1} T \int_0^t |\partial_t X - \partial_t x_h|^2 \,\mathrm{d}s \,\mathrm{d}\xi + Ch^2 \le Ch^2,
\end{aligned}$$

and the proof is completed. □

3.2 *FEM with tangential motions*

The aforementioned methods are developed based on the equation (1.1) and only normal motion is allowed. They might suffer from the fact that the mesh will have inhomogeneous properties during the evolution, for instance, some nodes may cluster and the mesh may become distorted. This will lead to instability and even the breakdown of the simulation. To address this challenge, various techniques have been proposed to improve the mesh quality for evolving various types of geometric flows in the literature, such as mesh redistribution (Bänsch *et al.*, 2005), and the introduction of artificial tangential velocity (Ševčovič & Mikula, 2001; Mikula & Ševčovič, 2004a; Barrett *et al.*, 2020; Duan & Li, 2024).

In this part, to achieve equipartition property for long-time evolution, we derive another formulation of (1.1) by introducing a tangential velocity. We consider the equation

$$\partial_t X = (\kappa - f(L))\mathscr{N} + \gamma(X)\mathscr{T},$$

where $\mathscr{N}, \mathscr{T}$ are the unit normal vector and tangent vector, respectively, and γ is the tangential velocity to be determined. It is important to note that the presence of tangential velocity has no impact on the shape of evolving curves (Deckelnick & Dziuk, 1995; Elliott & Fritz, 2017), and suitable choices of tangential velocity may help the redistribution of mesh points (Ševčovič & Mikula, 2001; Mikula & Ševčovič, 2004a,b; Kolár *et al.*, 2015). As mentioned in the introduction, inspired by the work of Deckelnick & Dziuk (1995); Elliott & Fritz (2017) for CSF, we consider an explicit tangential velocity given by

$$\gamma(X) = \frac{\partial_\xi X \cdot \partial_{\xi\xi} X}{|\partial_\xi X|^3}.$$

More generally, for a fixed parameter $0 < \alpha \leq 1$, we consider a series of reparametrizations X_α which are determined by

$$\alpha\partial_t X_\alpha + (1-\alpha)(\partial_t X_\alpha \cdot \mathscr{N})\mathscr{N} = \frac{\partial_{\xi\xi} X_\alpha}{|\partial_\xi X_\alpha|^2} - f(L)\mathscr{N}; \quad X_\alpha(\xi, 0) = X^0(\xi). \tag{3.16}$$

Below we provide three justifications for (3.16).

(i) The solution X_α of the evolution equation (3.16) has the same shape as the standard parametrization equation (1.2) since they share the same normal velocity

$$\partial_t X_\alpha \cdot \mathscr{N} = \alpha\partial_t X_\alpha \cdot \mathscr{N} + (1-\alpha)\partial_t X_\alpha \cdot \mathscr{N} = \left(\frac{\partial_{\xi\xi} X_\alpha}{|\partial_\xi X_\alpha|^2} - \gamma(X_\alpha)\mathscr{T} - f(L)\mathscr{N}\right) \cdot \mathscr{N}$$
$$= \left(\frac{1}{|\partial_\xi X_\alpha|}\partial_\xi\left(\frac{\partial_\xi X_\alpha}{|\partial_\xi X_\alpha|}\right) - f(L)\mathscr{N}\right) \cdot \mathscr{N} = \kappa - f(L),$$

where we note that the curvature κ and perimeter L are geometric quantities independent of the parametrization.

(ii) The evolution of (3.16) has asymptotic equidistribution property in a continuous level. More precisely, suppose X_α^e is the equilibrium of (3.16), i.e., $\partial_t X_\alpha^e = 0$, then formally we have

$$\partial_\xi |\partial_\xi X_\alpha^e| = \partial_{\xi\xi} X_\alpha^e \cdot \mathscr{T} = \left(\frac{\partial_{\xi\xi} X_\alpha^e}{|\partial_\xi X_\alpha^e|^2} - f(L)\mathscr{N} \right) \cdot |\partial_\xi X_\alpha^e|^2 \mathscr{T} = 0,$$

which means the equilibrium has constant arc-length. This leads us to expect that the corresponding numerical solution for (3.16) has equidistributed mesh points for long-time evolution.

(iii) As explained in Elliott & Fritz (2017, Section 8), we can write the standard parametrization equation (1.2) as

$$\partial_t X = \Delta_{\Gamma[X]} X - f(L)\mathscr{N},$$

where $\Gamma[X]$ is the image of X and $\Delta_{\Gamma[X]}$ is the Laplace–Beltrami operator over the curve $\Gamma[X]$. The DeTurck's trick for operator $\Delta_{\Gamma[X]}$ maintains the normal term $f(L)\mathscr{N}$ unaffected and leads to (3.16). In this aspect, the nonlocal flows can be viewed as a natural generalization of Elliott & Fritz (2017, Section 8).

Next we present an FEM for (3.16). For fixed α, multiplying $|\partial_\xi X|^2$ for both sides of (3.16) (below we omit the subscript α for simplicity), we obtain the following weak formulation: for any $v \in (H^1(\mathbb{S}^1))^2$, it holds

$$\int_{\mathbb{S}^1} |\partial_\xi X|^2 (\alpha \partial_t X + (1-\alpha)(\partial_t X \cdot \mathscr{N})\mathscr{N}) \cdot v \, \mathrm{d}\xi + \int_{\mathbb{S}^1} \partial_\xi X \cdot \partial_\xi v \, \mathrm{d}\xi + \int_{\mathbb{S}^1} f(L) |\partial_\xi X|^2 \mathscr{N} \cdot v \, \mathrm{d}\xi = 0. \tag{3.17}$$

We use the same spatial discretization for $\mathbb{S}^1$ as in the last subsection and assume it satisfies Assumption 3.1. We further assume the exact solution of (3.16) is regular in the following sense.

Assumption 3.3. Suppose that the solution of (3.16) with an initial value $X^0 \in H^2(\mathbb{S}^1)$ satisfies $X \in W^{1,\infty}\left([0,T], H^2(\mathbb{S}^1)\right)$, i.e.,

$$K_2(X) := \|X\|_{W^{1,\infty}([0,T],H^2(\mathbb{S}^1))} < \infty,$$

and there exist constants $0 < C_1 < C_2$ such that (2.2) holds.

Definition 3. We call a function $x_h(\xi,t) = \sum_{j=1}^{N} x_j(t)\varphi_j(\xi) : \mathbb{S}^1 \times [0,T] \to \mathbb{R}^2$ is a semidiscrete solution of (3.16) if it satisfies $x_h(\xi,0) = I_h X^0$ and

$$\int_{\mathbb{S}^1} |\partial_\xi x_h|^2 (\alpha \partial_t x_h + (1-\alpha)(\partial_t x_h \cdot n_h) n_h) \cdot v_h \, \mathrm{d}\xi + \int_{\mathbb{S}^1} \partial_\xi x_h \cdot \partial_\xi v_h \, \mathrm{d}\xi + \int_{\mathbb{S}^1} f(l_h) |\partial_\xi x_h|^2 n_h \cdot v_h \, \mathrm{d}\xi = 0, \tag{3.18}$$

for any $v_h \in V_h$, where $n_h = \tau_h^\perp$ represents the piecewise unit normal vector.

THEOREM 3. Let $X(\xi, t)$ be a solution of (3.16) satisfying Assumption 3.3. Assume that the partition of $\mathbb{S}^1$ satisfies Assumption 3.1. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$, there exists a unique semidiscrete solution x_h for (3.18). Furthermore, the solution satisfies

$$\sup_{t\in[0,T]} |X - x_h|^2_{H^1} + \alpha \int_0^T \|\partial_t X - \partial_t x_h\|^2_{L^2}\, dt + (1-\alpha)\int_0^T \|n_h \cdot (\partial_t X - \partial_t x_h)\|^2_{L^2}\, dt \le CTh^2 + Ce^{MT/\alpha}h^2,$$

where h_0, C and M depend on c_p, c_P, C_1, C_2, $K_2(X)$, and f.

We adapt the fixed-point argument (Deckelnick & Dziuk, 1995; Elliott & Fritz, 2017) and outline the key steps below.

Outline of proof. Fix $\alpha \in (0, 1]$. Consider the Banach space $Z_h = C([0, T], V_h)$ and a nonempty closed convex subset B_h (see the definition (3.19)) of Z_h. Define a continuous map $F : B_h \to Z_h$ by $F(u_h) = y_h$, where y_h is the solution of (3.22).

(1) The assumed estimate for B_h allows us to derive the length estimate for $u_h \in B_h$:

$$\|(|\partial_\xi X| - |\partial_\xi u_h|)(t)\|_{L^2} \le Khe^{Mt/(2\alpha)}.$$

(2) For any $y_h \in Z_h$, combining with Step (1), we can obtain the following stability estimate (cf. (3.27)):

$$\max_{t\in[0,T]} \|\partial_\xi X - \partial_\xi y_h\|^2_{L^2} + \alpha \int_0^T \|\partial_t X - \partial_t y_h\|^2_{L^2}\, dt + (1-\alpha)\int_0^T \|\widehat{n_h} \cdot (\partial_t X - \partial_t y_h)\|^2_{L^2}\, dt$$
$$\le CTh^2 + Ce^{MT/\alpha}h^2.$$

Finally, we use the above estimate to show that $F(B_h) \subset B_h$ and apply Schauder's fixed-point theorem to obtain a fixed point x_h, which is the desired solution and satisfies the corresponding estimate. □

Proof of Theorem 3. Fix $\alpha \in (0, 1]$. We consider a Banach space $Z_h = C([0, T], V_h)$ equipped with the norm

$$\|v_h\|_{Z_h} := \sup_{t\in[0,T]} \|v_h(t)\|_{L^2}, \quad v_h \in Z_h,$$

and a nonempty closed convex subset B_h of Z_h defined by

$$B_h := \left\{ v_h \in Z_h | \sup_{t\in[0,T]} e^{-Mt/\alpha} \|(\partial_\xi X - \partial_\xi v_h)(t)\|^2_{L^2} \le K^2h^2 \text{ and } v_h(\cdot, 0) = I_h X^0(\cdot) \right\}, \tag{3.19}$$

where $M, K > 0$ are constants that will be determined later. For any $u_h \in B_h$, applying interpolation error, inverse inequality and (3.19), one can easily derive

$$\begin{aligned}
\|(\partial_\xi X - \partial_\xi u_h)(t)\|_{L^\infty} &\le \|(\partial_\xi X - I_h\partial_\xi X)(t)\|_{L^\infty} + \|(I_h\partial_\xi X - \partial_\xi I_h u_h)(t)\|_{L^\infty}\\
&\le Ch^{1/2} + Ch^{-1/2}\|(I_h\partial_\xi X - \partial_\xi u_h)(t)\|_{L^2}\\
&\le Ch^{1/2} + Ch^{-1/2}\left(\|(\partial_\xi X - I_h\partial_\xi X)(t)\|_{L^2} + \|(\partial_\xi X - \partial_\xi u_h)(t)\|_{L^2}\right)\\
&\le Ch^{1/2}\left(1 + e^{\frac{Mt}{2\alpha}}K\right).
\end{aligned}$$

It follows from Assumption 3.3 that there exists a constant $h_0 > 0$, depending on $\alpha, M, K, T, K_2(X)$ such that for any $0 < h \le h_0$, we have

$$\inf_{\xi} |\partial_\xi u_h| \ge C_1/2, \quad \sup_{\xi} |\partial_\xi u_h| \le 2C_2. \tag{3.20}$$

Setting $\widehat{q_h} = |\partial_\xi u_h|$ and denoting $\widehat{l_h}$ as the perimeter of u_h, due to the Lipschitz property of f, it holds that

$$2\pi C_1 \le L \le 2\pi C_2, \quad \pi C_1 \le \widehat{l_h} \le 4\pi C_2, \quad |f(\widehat{l_h})| \le C, \tag{3.21}$$

where C is a constant, depending on C_1, C_2 and f. We define a continuous map $F : B_h \to Z_h$ as follows. For any $u_h \in B_h$, we define y_h as the unique solution which satisfies

$$\int_{\mathbb{S}^1} \widehat{q_h}^2(\alpha\partial_t y_h + (1-\alpha)(\partial_t y_h \cdot \widehat{n_h})\widehat{n_h}) \cdot v_h \,\mathrm{d}\xi + \int_{\mathbb{S}^1} \partial_\xi y_h \cdot \partial_\xi v_h \,\mathrm{d}\xi + \int_{\mathbb{S}^1} f(\widehat{l_h})\widehat{q_h}(\partial_\xi y_h)^\perp \cdot v_h \,\mathrm{d}\xi = 0, \tag{3.22}$$

for all $v_h \in V_h$, with initial data $y_h(0) = I_h X^0$, where $\widehat{n_h} = \left(\frac{\partial_\xi u_h}{|\partial_\xi u_h|}\right)^\perp$.

(1) *Length difference estimate for $u_h \in B_h$*. Applying (3.19) and the triangle inequality, we obtain

$$\|(|\partial_\xi X| - \widehat{q_h})(t)\|_{L^2} \le \|(\partial_\xi X - \partial_\xi u_h)(t)\|_{L^2} \le Khe^{Mt/(2\alpha)}, \quad 0 \le t \le T. \tag{3.23}$$

(2) *Stability estimate for $y_h \in Z_h$*. Taking $v = v_h$ in (3.17) and subtracting (3.22) from (3.17), we get

$$\begin{aligned}
&\int_{\mathbb{S}^1} \widehat{q_h}^2 \left(\alpha \left(\partial_t X - \partial_t y_h\right) \cdot v_h + (1-\alpha) \left(\partial_t X - \partial_t y_h\right) \cdot \widehat{n_h}(\widehat{n_h} \cdot v_h)\right) \,\mathrm{d}\xi + \int_{\mathbb{S}^1} \left(\partial_\xi X - \partial_\xi y_h\right) \cdot \partial_\xi v_h \,\mathrm{d}\xi \\
&= \int_{\mathbb{S}^1} (\widehat{q_h}^2 - |\partial_\xi X|^2)(\alpha\partial_t X \cdot v_h + (1-\alpha)(\partial_t X \cdot \widehat{n_h})(\widehat{n_h} \cdot v_h)) \,\mathrm{d}\xi \\
&\quad + (1-\alpha) \int_{\mathbb{S}^1} |\partial_\xi X|^2 (\partial_t X \cdot (\widehat{n_h} - \mathcal{N})(\widehat{n_h} \cdot v_h) + (\partial_t X \cdot \mathcal{N})(\widehat{n_h} - \mathcal{N}) \cdot v_h) \,\mathrm{d}\xi \\
&\quad + \int_{\mathbb{S}^1} \left(-f(L)|\partial_\xi X| + f(\widehat{l_h})\widehat{q_h}\right) (\partial_\xi X)^\perp \cdot v_h \,\mathrm{d}\xi \\
&\quad - \int_{\mathbb{S}^1} f(\widehat{l_h})\widehat{q_h} \left((\partial_\xi X)^\perp - (\partial_\xi y_h)^\perp\right) \cdot v_h \,\mathrm{d}\xi =: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Choosing $v_h = I_h(\partial_t X) - \partial_t y_h \in V_h$, the estimates of the left-hand side and J_1, J_2 can be found in Elliott & Fritz (2017, (3.7)), which can be summarized as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_\xi X - \partial_\xi y_h\|_{L^2}^2 + \frac{C_1^2}{8}\alpha\|\partial_t X - \partial_t y_h\|_{L^2}^2 + \frac{C_1^2}{4}(1-\alpha)\|\widehat{n_h} \cdot (\partial_t X - \partial_t y_h)\|_{L^2}^2 \\
&\le Ch^2(1 + \|\partial_t X\|_{H^2}^2) + \|\partial_\xi X - \partial_\xi y_h\|_{L^2}^2 + Ch^2K^2e^{Mt/\alpha}/\alpha + 2|J_3| + 2|J_4|.
\end{aligned} \tag{3.24}$$

For the terms of J_3 and J_4, in view of the Lipschitz property of f and the inequality

$$|L-\widehat{l_h}| \le C\||\partial_\xi X|-\widehat{q_h}\|_{L^2},$$

applying (3.20), (3.21), (3.23) and Young's inequality, we get

$$\begin{aligned}
|J_3| &= \left|\int_{\mathbb{S}^1}\left(-f(L)+f(\widehat{l_h})\right)|\partial_\xi X|(\partial_\xi X)^\perp\cdot v_h\,\mathrm{d}\xi - \int_{\mathbb{S}^1} f(\widehat{l_h})\left(|\partial_\xi X|-\widehat{q_h}\right)(\partial_\xi X)^\perp\cdot v_h\,\mathrm{d}\xi\right|\\
&\le C\int_{\mathbb{S}^1}|L-\widehat{l_h}|\left(|I_h(\partial_t X)-\partial_t X|+|\partial_t X-\partial_t y_h|\right)\,\mathrm{d}\xi\\
&\quad + C\int_{\mathbb{S}^1}||\partial_\xi X|-\widehat{q_h}|\left(|I_h(\partial_t X)-\partial_t X|+|\partial_t X-\partial_t y_h|\right)\,\mathrm{d}\xi\\
&\le Ch|L-\widehat{l_h}|\|\partial_t X\|_{H^1}+C|L-\widehat{l_h}|\|\partial_t X-\partial_t y_h\|_{L^2}\\
&\quad + Ch\||\partial_\xi X|-\widehat{q_h}\|_{L^2}\|\partial_t X\|_{H^1}+C\||\partial_\xi X|-\widehat{q_h}\|_{L^2}\|\partial_t X-\partial_t y_h\|_{L^2}\\
&\le CKe^{\frac{Mt}{2\alpha}}h\|\partial_t X-\partial_t y_h\|_{L^2}+CKe^{\frac{Mt}{2\alpha}}h^2\|\partial_t X\|_{H^1}\\
&\le \frac{C(\varepsilon)e^{Mt/\alpha}K^2h^2}{4\alpha}+\varepsilon\alpha\|\partial_t X-\partial_t y_h\|_{L^2}^2+\alpha h^2\|\partial_t X\|_{H^1}^2,
\end{aligned}$$

and

$$\begin{aligned}
|J_4| &= \left|\int_{\mathbb{S}^1} f(\widehat{l_h})\widehat{q_h}\left((\partial_\xi X)^\perp-(\partial_\xi y_h)^\perp\right)\cdot v_h\,\mathrm{d}\xi\right|\\
&\le Ch\|\partial_\xi X-\partial_\xi y_h\|_{L^2}\|\partial_t X\|_{H^1}+C\|\partial_\xi X-\partial_\xi y_h\|_{L^2}\|\partial_t X-\partial_t y_h\|_{L^2}\\
&\le \|\partial_\xi X-\partial_\xi y_h\|_{L^2}^2+Ch^2\|\partial_t X\|_{H^1}^2+\frac{C(\varepsilon)}{4\alpha}\|\partial_\xi X-\partial_\xi y_h\|_{L^2}^2+\varepsilon\alpha\|\partial_t X-\partial_t y_h\|_{L^2}^2.
\end{aligned}$$

Combining all the above estimate and taking ε small enough, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_\xi X-\partial_\xi y_h\|_{L^2}^2+\frac{C_1^2}{16}\alpha\|\partial_t X-\partial_t y_h\|_{L^2}^2+\frac{C_1^2}{4}(1-\alpha)\|\widehat{n_h}\cdot(\partial_t X-\partial_t y_h)\|_{L^2}^2\\
&\le\ Ch^2+C(1+1/\alpha)\|\partial_\xi X-\partial_\xi y_h\|_{L^2}^2+Ch^2K^2e^{Mt/\alpha}/\alpha, \qquad (3.25)
\end{aligned}$$

where C depends on c_p, c_P, C_1, C_2, T, $K_2(X)$ and f. This directly gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_\xi X-\partial_\xi y_h\|_{L^2}^2 \le\ Ch^2+C(1+1/\alpha)\|\partial_\xi X-\partial_\xi y_h\|_{L^2}^2+Ch^2K^2e^{Mt/\alpha}/\alpha.$$

Thus, we get

$$\|\partial_\xi X(t) - \partial_\xi y_h(t)\|_{L^2}^2 \le \|\partial_\xi X^0 - \partial_\xi y_h(0)\|_{L^2}^2 e^{C(1+\frac{1}{\alpha})t} + Ch^2 \int_0^t e^{C(1+\frac{1}{\alpha})(t-s)} \left(1 + K^2 e^{Ms/\alpha}/\alpha\right) \mathrm{d}s$$
$$\le Ce^{C(1+\frac{1}{\alpha})t} h^2 + CK^2h^2 \frac{e^{\frac{M}{\alpha}t} - e^{C(1+\frac{1}{\alpha})t}}{M - C(1+\alpha)},$$

which yields

$$e^{-Mt/\alpha} \|\partial_\xi X - \partial_\xi y_h\|_{L^2}^2 \le Ch^2 e^{(-\frac{M}{\alpha}+\frac{C}{\alpha}+C)t} + \frac{CK^2h^2}{M - C(1+\alpha)} \le K^2h^2, \tag{3.26}$$

if we select $M \ge 3C + C\alpha$ and $K^2 \ge 2C$.

Hence, by plugging (3.26) into (3.25), integrating from 0 to T, we arrive at

$$\max_{t\in[0,T]} \|\partial_\xi X - \partial_\xi y_h\|_{L^2}^2 + \alpha \int_0^T \|\partial_t X - \partial_t y_h\|_{L^2}^2 \,\mathrm{d}t + (1-\alpha) \int_0^T \|\widehat{n_h} \cdot (\partial_t X - \partial_t y_h)\|_{L^2}^2 \,\mathrm{d}t$$
$$\le C(1+T)h^2 + CK^2h^2(1+1/\alpha) \int_0^T e^{Ms/\alpha} \,\mathrm{d}s$$
$$\le C(1+T)h^2 + K^2h^2(e^{MT/\alpha} - 1) \le CTh^2 + Ce^{MT/\alpha}h^2, \tag{3.27}$$

where C and M are constants, depending on $c_p, c_P, K_2(X), C_1, C_2$ and f.

Now we complete the proof by applying Schauder's fixed point theorem for F. Indeed, it follows from assumption (3.19) and (3.26) that $F(B_h) \subset B_h$. Furthermore, it can be easily derived from (3.27) and the assumption $y_h(0) = I_h X^0$ that $\|y_h\|_{W^{1,2}([0,T],V_h)} \le C$, which, together with the Sobolev embedding, implies that the inclusion $F(B_h) \subset B_h$ is compact. Thus, by Schauder's fixed point theorem (c.f. (Elliott & Fritz, 2017, Theorem 3.1)), there exists a fixed point x_h for (3.22) that satisfies $F(x_h) = x_h$, which is the desired semidiscrete solution. Moreover, the estimate (3.27) also holds for the solution x_h.

To address the uniqueness, it is important to recognize that (3.18) constitutes a nonlinear ODE system for x_j. Consequently, the uniqueness of x_h is assured by nonlinear ODE theory. It's evident that x_h serves as a semidiscrete solution of (3.18), and thus aligns with the corresponding estimate. □

4. Convergence under manifold distance

As discussed in Zhao *et al.* (2021); Jiang *et al.* (2024a), for two closed simple curves Γ_1 and Γ_2, the manifold distance is defined as

$$\mathrm{M}\left(\Gamma_1, \Gamma_2\right) := \mathrm{Area}((\Omega_1 \setminus \Omega_2) \cup (\Omega_2 \setminus \Omega_1)) = \mathrm{Area}(\Omega_1) + \mathrm{Area}(\Omega_2) - 2\mathrm{Area}(\Omega_1 \cap \Omega_2),$$

where Ω_1 and Ω_2 are the regions enclosed by Γ_1 and Γ_2, respectively. As proved in Zhao *et al.* (2021, Proposition 5.1), the manifold distance satisfies symmetry, positivity and the triangle inequality. Under some assumptions, e.g., if Γ_2 lies within the tabular neighborhood of the C^2 curve Γ_1 (Deckelnick *et al.*, 2005), the manifold distance between the two curves can be interpreted as the L^1-norm of the distance function. Recently, compared to the L^p-norm of parametrization functions, this type of distance (i.e. the L^p-norm of distance function) has gained wide attention in both the scientific computing (Jiang *et al.*, 2024a,b) and numerical analysis communities (Bai & Li, 2023). Moreover, the authors' works (Zhao *et al.*, 2021; Jiang *et al.*, 2024a,b) have demonstrated that the manifold distance (one of the shape metrics) is more suitable than the norm of parametrization functions for quantifying numerical errors of the schemes which are used for solving geometric flows, especially for schemes which allow intrinsic tangential velocity. Meanwhile, Bai and Li (Bai & Li, 2023) have recently observed that the L^2-norm of distance function (so-called the projected distance in their paper) leads to the recovery of full H^1 parabolicity, and established a convergence result for Dziuk's scheme of the mean curvature flow with finite elements of degree $k \geq 3$.

In this subsection, we first show that the function L^∞-norm is stronger than the manifold distance under some suitable regularity assumptions. More specifically, for a parametrization function $X \in C^2(\mathbb{S}^1)$ of curve Γ_X and an approximation curve Γ_Y by parameterization function $Y \in C^0(\mathbb{S}^1)$, we have the following lemma.

LEMMA 4. Let $X : \mathbb{S}^1 \to \mathbb{R}^2$ be a parametrization function of simple curves Γ_X with $X \in C^2(\mathbb{S}^1)$. Assume there exist constants $0 < C_1 < C_2$ such that it holds

$$C_1 \leq \left|\partial_\xi X(\xi)\right| \leq C_2, \quad \forall\, \xi \in \mathbb{S}^1.$$

Then there exist positive constants δ_0 and C such that for any parametrization function $Y \in C^0(\mathbb{S}^1)$ satisfying

$$\|X - Y\|_{L^\infty} \leq \delta_0,$$

the following inequality is true:

$$\mathrm{M}(\Gamma_X, \Gamma_Y) \leq C\|X - Y\|_{L^\infty},$$

where Γ_X and Γ_Y are the images of X and Y, respectively. The constants δ_0 and C depend on X, C_1 and C_2.

Proof. The closed simple C^2 curve Γ_X in $\mathbb{R}^2$ naturally admits a tabular neighborhood Ω_δ in the following manner (Deckelnick *et al.*, 2005; Bänsch *et al.*, 2023): there exists a constant $\delta > 0$ such that the mapping

$$E_X : \Gamma_X \times (-\delta, \delta) \to \mathbb{R}^2, \quad E_X(a, \eta) = a + \eta\mathscr{N},$$

acts as a diffeomorphism from $\Gamma_X \times (-\delta, \delta)$ to the image denoted by $\mathrm{Im}(E_X) =: \Omega_\delta$. Here $\mathscr{N}$ represents the normal vector along Γ_X. Consequently, the points within the tabular neighborhood Ω_δ can be represented as

$$E_X^{-1} : \Omega_\delta \to \Gamma_X \times (-\delta, \delta), \quad E_X^{-1}(b) = (\pi_{\Gamma_X}(b), d_{\Gamma_X}(b)),$$

where $\pi_{\Gamma_X}(b) \in \Gamma_X$ is the projection of b onto Γ_X, and $d_{\Gamma_X}(b) = d(b, \Gamma_X)$ represents the signed distance.

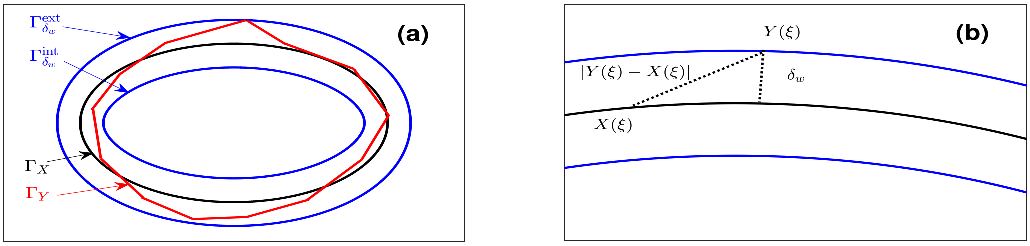

FIG. 1. Illustration of (a) the definition of $\Gamma_X, \Gamma_Y, \Gamma_{\delta_w}^{\text{int}}$ and $\Gamma_{\delta_w}^{\text{ext}}$, (b) the comparison of the projection distance δ_w and the function L^∞-norm $\|X - Y\|_{L^\infty}$.

Set $\delta_0 < \delta$. For any parametrization function $Y \in C^0(\mathbb{S}^1)$ which satisfies $\|X - Y\|_{L^\infty} \le \delta_0$, it is evident that $\Gamma_Y \subseteq \Omega_\delta$. Now define

$$\delta_w := \sup_{b \in \Gamma_Y} |d_{\Gamma_X}(b)|$$

which represents the maximum distance between Γ_Y and Γ_X. Clearly, we can assume $\delta_w > 0$, as there is nothing to prove otherwise. Define two curves $\Gamma_{\delta_w}^{\text{int}}$ and $\Gamma_{\delta_w}^{\text{ext}}$, within the tabular neighborhood Ω_δ, which are parametrized as

$$\begin{aligned}
&\mathbb{S}^1 \ni \xi \to (x_{\text{int}}(\xi), y_{\text{int}}(\xi)) = (x(\xi), y(\xi)) + \delta_w \mathcal{N}(x(\xi), y(\xi)), \\
&\mathbb{S}^1 \ni \xi \to (x_{\text{ext}}(\xi), y_{\text{ext}}(\xi)) = (x(\xi), y(\xi)) - \delta_w \mathcal{N}(x(\xi), y(\xi)),
\end{aligned} \tag{4.1}$$

respectively. Here $X(\xi) = (x(\xi), y(\xi))$ is a parametrization of the curve Γ_X, and $\mathcal{N}$ is the unit inner normal vector. Denote Ω_{δ_w} as the region enclosed by $\Gamma_{\delta_w}^{\text{int}}$ and $\Gamma_{\delta_w}^{\text{ext}}$ (cf. Fig. 1(a)). By utilizing the regularity assumption of X along with (2.2) and (4.1), we can estimate the area of Ω_{δ_w} as follows:

$$\begin{aligned}
\text{Area}(\Omega_{\delta_w}) &= \int_{\mathbb{S}^1} \partial_\xi x_{\text{ext}}\, y_{\text{ext}}\, \mathrm{d}\xi - \int_{\mathbb{S}^1} \partial_\xi x_{\text{int}}\, y_{\text{int}}\, \mathrm{d}\xi \\
&\le C \int_{\mathbb{S}^1} |\partial_\xi x_{\text{ext}} - \partial_\xi x_{\text{int}}| + |y_{\text{ext}} - y_{\text{int}}|\, \mathrm{d}\xi \\
&\le C\delta_w \int_{\mathbb{S}^1} |\partial_\xi \mathcal{N}| + |\mathcal{N}|\, \mathrm{d}\xi \le C\delta_w,
\end{aligned}$$

where C is a constant, depending on X, C_1 and C_2. The triangle inequality for manifold distance yields

$$\mathrm{M}(\Gamma_X, \Gamma_Y) \le \mathrm{M}(\Gamma_X, \Gamma_{\delta_w}^{\text{int}}) + \mathrm{M}(\Gamma_{\delta_w}^{\text{int}}, \Gamma_Y) \le 2\text{Area}(\Omega_{\delta_w}) \le C\delta_w \le C\|X - Y\|_{L^\infty},$$

where we use the natural control $\delta_w \le \|X - Y\|_{L^\infty}$ (cf. Fig. 1(b)) and the proof is completed. $\square$

As natural corollaries, we have the following convergence results of numerical schemes under the manifold distance.

COROLLARY 1.

(1) Let $X(\xi, t)$ be the solution of (1.2) that satisfies Assumption 2.1 and belongs to $W^{3,2}(\mathbb{S}^1)$. Let $x_h(t)$ be the unique finite difference semidiscrete solution of (2.6), then it holds

$$\sup_{t\in[0,T]} \mathrm{M}(\Gamma_X, \Gamma_{x_h}) \le Ch^2.$$

(2) Let $X(\xi, t)$ be the solution of (1.2) that satisfies Assumption 3.2 and belongs to $W^{3,2}(\mathbb{S}^1)$. Assume that the partition of $\mathbb{S}^1$ satisfies Assumption 3.1, and let $x_h(t)$ be the unique finite element semidiscrete solution of (3.3). Then, it holds

$$\sup_{t\in[0,T]} \mathrm{M}(\Gamma_X, \Gamma_{x_h}) \le Ch.$$

(3) Let $X(\xi, t)$ be the solution of (3.16) that satisfies Assumption 3.3 and belong to $W^{3,2}(\mathbb{S}^1)$. Assume that the partition of $\mathbb{S}^1$ satisfies Assumption 3.1 and $x_h(t)$ is the unique finite element semidiscrete solution of (3.18), then

$$\sup_{t\in[0,T]} \mathrm{M}(\Gamma_X, \Gamma_{x_h}) \le Ch.$$

In all estimates, Γ_X and Γ_{x_h} represent the images of X and x_h, respectively. The constant C depends on C_1, C_2, T, f and additionally, $K_1(X)$ for (1), and $K_2(X)$ for (2) and (3).

Proof. For the first conclusion, combining the Sobolev embedding, triangle inequality, interpolation error, Lemma 5 with the main error estimate (2.7), one obtains

$$\begin{aligned}\|X - x_h\|_{L^\infty} &\le C\|X - x_h\|_{H^1} \le C\|X - I_hX\|_{H^1} + C\|I_hX - x_h\|_{H^1}\\ &\le Ch^2\|X\|_{W^{3,2}} + C\|I_hX - x_h\|_{H^1_G}\sqrt{1 + h^2/6} \le Ch^2,\end{aligned}$$

where for the third inequality we have utilized Lemma 5 for the grid function $I_hX - x_h$. Hence, by applying Lemma 4 with $\delta_0 = Ch^2$, $Y = x_h$ and for different time $t \in [0, T]$, we conclude the first assertion (1). The latter two statements can be similarly confirmed by referring to Theorems 2 and 3, along with the Sobolev embedding

$$\sup_{t\in[0,T]} \|X - x_h\|_{L^\infty} \le C \sup_{t\in[0,T]} \|X - x_h\|_{H^1} \le Ch,$$

and the proof is completed. □

LEMMA 5. Let $g : \mathscr{G}_h \to \mathbb{R}$ be a grid function. Then it holds

$$\|g\|^2_{H^1} \le \|g\|^2_{H^1_G}\left(1 + h^2/6\right),$$

where the grid function g is identified with the piecewise linear function over $\mathbb{S}^1$ that connects the grid values of g.

Proof. Denote $M_j = \max_{\xi\in(\xi_j,\xi_{j+1})} \left(|g|^2\right)''$ and by applying the trapezoidal rule, we have

$$
\begin{aligned}
\|g\|_{H^1}^2 &= \sum_{j=1}^{N} \int_{\xi_j}^{\xi_{j+1}} |g|^2 + |\partial_\xi g|^2 \,\mathrm{d}\xi \\
&\le \sum_{j=1}^{N} \left(\frac{|g(\xi_j)|^2 + |g(\xi_{j+1})|^2}{2} h + \frac{h^3}{12} M_j \right) + h \sum_{j=1}^{N} |\delta g_{j+1}|^2 \\
&= h \sum_{j=1}^{N} \left(|g_j|^2 + |\delta g_j|^2\right) + \frac{h^3}{12} \sum_{j=1}^{N} M_j.
\end{aligned}
$$

Noticing g is a piecewise linear function, we have

$$
M_j = \max_{\xi\in(\xi_j,\xi_{j+1})} \left(|g|^2\right)'' = 2 \max_{\xi\in(\xi_j,\xi_{j+1})} |g'|^2 = 2|\delta g_{j+1}|^2,
$$

which yields

$$
\|g\|_{H^1}^2 \le h \sum_{j=1}^{N} \left(|g_j|^2 + |\delta g_j|^2\right) + \frac{h^3}{6} \sum_{j=1}^{N} |\delta g_{j+1}|^2 \le \|g\|_{H_G^1}^2 \left(1 + \frac{1}{6} h^2\right),
$$

and the proof is completed. □

5. Numerical results

In this section, we present numerous numerical experiments for the proposed three different schemes applied to various geometric flows involving the nonlocal term $f(L)$. We first provide full discretizations for the three schemes using backward Euler time discretization. Specifically, we choose an integer m, set the time step $\tau = T/m$ and $t_k = k\tau$ for $k = 0, \ldots, m$. Given a fixed mesh size h and a time step $\tau = O(h^2)$, we consider the following three cases.

(i) For the finite difference method (2.6), given $x_h^0 = I_h X^0$, for $k \ge 1$, we consider the solution $x_h^k \in \mathscr{G}_h$ of the following equation (**denoted as FDM**)

$$
\frac{x_j^k - x_j^{k-1}}{\tau} = \frac{2}{q_j^{k-1} + q_{j+1}^{k-1}} \left(\frac{x_{j+1}^k - x_j^k}{q_{j+1}^{k-1}} - \frac{x_j^k - x_{j-1}^k}{q_j^{k-1}} \right) - f(l_h^{k-1}) \frac{n_j^{k-1} + n_{j+1}^{k-1}}{|n_j^{k-1} + n_{j+1}^{k-1}|}, \tag{5.1}
$$

where x_j^k represents the grid value, l_h^k is the perimeter of the polygon with vertices $\{x_j^k\}_j$, and $n_j^k = (\tau_j^k)^\perp$ is the discrete normal vector.

(ii) For the finite element method (3.3), given $x_h^0 = I_h X^0$, for $k \geq 1$, we consider the solution $x_h^k = \sum_{j=1}^{N} x_j^k \varphi_j \in V_h$ which satisfies (**denoted as FEM**)

$$\int_{\mathbb{S}^1} \left|\partial_\xi x_h^{k-1}\right| \delta_\tau x_h^k \cdot v_h \,\mathrm{d}\xi + \int_{\mathbb{S}^1} \partial_\xi x_h^k \cdot \partial_\xi v_h / \left|\partial_\xi x_h^{k-1}\right| \,\mathrm{d}\xi$$
$$+ \int_{\mathbb{S}^1} \mathbf{h}^2 |\partial_\xi x_h^{k-1}| \partial_\xi \delta_\tau x_h^k \cdot \partial_\xi v_h / 6 \,\mathrm{d}\xi + \int_{\mathbb{S}^1} f\left(l_h^{k-1}\right)\left(\partial_\xi x_h^k\right)^\perp \cdot v_h \,\mathrm{d}\xi = 0, \quad \forall\, v_h \in V_h,$$

where δ_τ is the backward finite difference $\delta_\tau x_h^k = (x_h^k - x_h^{k-1})/\tau$, and l_h^{k-1} is the length of the image of x_h^{k-1}. Or it can be written equivalently as a discretization for the ODE system (3.5):

$$\frac{q_j^{k-1} + q_{j+1}^{k-1}}{2\tau}\left(x_j^k - x_j^{k-1}\right) - \frac{x_{j+1}^k - x_j^k}{q_{j+1}^{k-1}} + \frac{x_j^k - x_{j-1}^k}{q_j^{k-1}} + \frac{f\left(l_h^{k-1}\right)}{2}\left(x_{j+1}^k - x_{j-1}^k\right)^\perp = 0. \tag{5.2}$$

(iii) For the finite element method with tangent motions (3.18), given $x_h^0 = I_h X^0$, for fixed $\alpha \in (0, 1]$ and $k \geq 1$, $x_h^k = \sum_{j=1}^{N} x_j^k \varphi_j \in V_h$ is the solution of the following (**denoted as FEM-TM**)

$$\int_{\mathbb{S}^1} I_h\left[(\alpha\delta_\tau x_h^k + (1-\alpha)(\delta_\tau x_h^k \cdot n_h^{k-1}) n_h^{k-1}) \cdot v_h\right] \left|\partial_\xi x_h^{k-1}\right|^2 \mathrm{d}\xi + \int_{\mathbb{S}^1} \partial_\xi x_h^k \cdot \partial_\xi v_h \,\mathrm{d}\xi$$
$$+ \int_{\mathbb{S}^1} I_h\left[f(l_h^{k-1}) n_h^{k-1} \cdot v_h\right] \left|\partial_\xi x_h^{k-1}\right|^2 \mathrm{d}\xi = 0, \quad \forall\, v_h \in V_h, \tag{5.3}$$

where $n_h^{k-1} = \left(\frac{\partial_\xi x_h^{k-1}}{|\partial_\xi x_h^{k-1}|}\right)^\perp$ is the unit normal vector. Through a straightforward computation, we find it can be written equivalently as

$$\alpha \frac{x_j^k - x_j^{k-1}}{\tau} + (1-\alpha)\left(\frac{x_j^k - x_j^{k-1}}{\tau} \cdot n_j^{k-1}\right) n_j^{k-1} = \frac{2(x_{j+1}^k - 2x_j^k + x_{j-1}^k)}{\left(q_j^{k-1}\right)^2 + \left(q_{j+1}^{k-1}\right)^2} - f\left(l_h^{k-1}\right) n_j^{k-1}, \tag{5.4}$$

where $n_j^{k-1} = \left(\frac{x_j^{k-1} - x_{j-1}^{k-1}}{q_j^{k-1}}\right)^\perp$.

REMARK 2. The convergence analysis of the fully discrete schemes, including FEM (5.1), FEM (5.2) and FEM-TM (5.4), requires considerable investigation. It is noteworthy that the FDM and FEM schemes are natural extensions of Dziuk's fully discrete linearly implicit scheme for CSF (Dziuk, 1994), incorporating an additional nonlocal forcing term. Recent advancements in error estimates for Dziuk-type schemes include: (i) Ye & Cui (2021), who employed an unconditional length shortening property ($q_j^k \leq q_j^{k-1}$ for any $\tau > 0$) and matrix analysis techniques to establish an optimal error estimate in the H^1-norm for Dziuk's lumped mass scheme; (ii) Li (2020), who identified a monotone structure and examined convergence results for Dziuk's original scheme with higher-order finite elements ($k \geq 3$); and (iii) Bai & Li (2023), who introduced a new framework to prove the convergence of Dziuk's original scheme with higher-order finite elements ($k \geq 3$) under the projected distance. However, extending these approaches to the nonlocal case is challenging due to the loss of unconditional length shortening property and the failure of monotone structure in our formulation. Moreover, extending the framework proposed by Bai and Li is significantly more complex. The difficulty with Dziuk-type schemes stems from their lack of parabolicity (cf. Li (2020); Ye & Cui (2021)). Incorporating tangential velocity enhances parabolicity and simplifies numerical analysis (Deckelnick & Dziuk, 1995; Barrett *et al.*, 2017; Elliott & Fritz, 2017). Utilizing techniques from the work of Barrett, Deckelnick, and Styles (Barrett *et al.*, 2017), we can establish a convergence result for the FEM-TM scheme (5.3). Detailed proofs can be found in the appendix for the readers' convenience.

5.1 *Accuracy test*

To evaluate the convergence order of the proposed three schemes, we primarily consider the following cases of geometric flows with different initial curves:

Case 1: An ellipse initial curve, parameterized by $(2\cos\theta, \sin\theta)^T$, $\theta \in [0, 2\pi]$, with the corresponding flow being the AP-CSF with $f(L) = 2\pi/L$;

Case 2: A four-leaf rose initial curve, parameterized by $(\cos(2\theta)\cos\theta, \cos(2\theta)\sin\theta)^T$, $\theta \in [0, 2\pi]$, with the corresponding flow being the AP-CSF for nonsimple curves with $f(L) = 2\pi\,\mathrm{ind}/L$, $\mathrm{ind}(\Gamma) = 3$;

Case 3: An ellipse initial curve, with the corresponding flow being a curve flow with area decreasing rate of π, i.e., $f(L) = (2\pi - \beta)/L$, $\beta = \pi$.

As the exact solutions of the above cases are unknown, we consider the following numerical errors for the FDM (5.1):

$$L_t^\infty H_G^1 \text{ error} \qquad (\mathscr{E}_1)_{h,\tau}(T) := \max_{1\leq k\leq T/\tau} \left\| x_{h,\tau}^k - \widehat{x}_{h/2,\tau/4}^{4k} \right\|_{H_G^1},$$

$$\text{Manifold distance} \quad (\mathscr{E}_2)_{h,\tau}(T) := \mathrm{M}\left(\Gamma_{h,\tau}^{T/\tau}, \Gamma_{h/2,\tau/4}^{4T/\tau}\right),$$

where we view $\widehat{x}_{h/2,\tau/4}^{4k}$ as a grid function over $\mathscr{G}_h$ with grid values $\{x_{2j}^{4k}\}_{j=1}^N$, and the L_G^∞ norm is defined as $\|g\|_{L_G^\infty} := \max_{j=1,\ldots,N} |g_j|$. Furthermore, the polygons $\Gamma_{h,\tau}^{T/\tau}$ and $\Gamma_{h/2,\tau/4}^{4T/\tau}$ are the images of $x_{h,\tau}^{T/\tau}$ and $x_{h/2,\tau/4}^{4T/\tau}$, respectively.

Different types of errors for the FDM (5.1) are depicted Fig. 2, where we choose $h = 2\pi/N, \tau = 0.5h^2$. The numerical results indicate that, for each instance of nonlocal flows listed above, the solution of (5.1) converges quadratically in $L_t^\infty L_G^2$, $L_t^\infty H_G^1$ and $L_t^\infty L_G^\infty$, which agrees with the theoretical results in Theorem 1. Moreover, we observe a quadratic convergence under the manifold distance, aligning with the theoretical findings in Corollary 1 (1).

We now turn to the convergence order test of the FEM (5.2) and the FEM-TM (5.4). We similarly consider the following numerical errors

$$L_t^\infty H_x^1 \text{ error} \qquad (\mathscr{E}_3)_{h,\tau}(T) := \max_{1\leq k\leq T/\tau} \left(\left\| x_{h,\tau}^k - x_{h/2,\tau/4}^{4k} \right\|_{L^2(\mathbb{S}^1)} + \left\| \partial_\xi x_{h,\tau}^k - \partial_\xi x_{h/2,\tau/4}^{4k} \right\|_{L^2(\mathbb{S}^1)} \right),$$

$$\text{Manifold distance} \quad (\mathscr{E}_4)_{h,\tau}(T) := \mathrm{M}(\Gamma_{h,\tau}^{T/\tau}, \Gamma_{h/2,\tau/4}^{4T/\tau}),$$

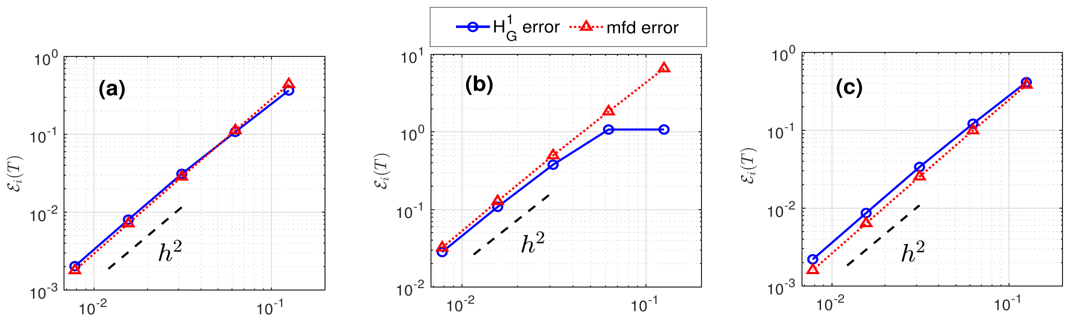

FIG. 2. Numerical errors under different norms for the FDM (5.1) at $T = 1/4$: (a) Case 1; (b) Case 2; (c) Case 3.

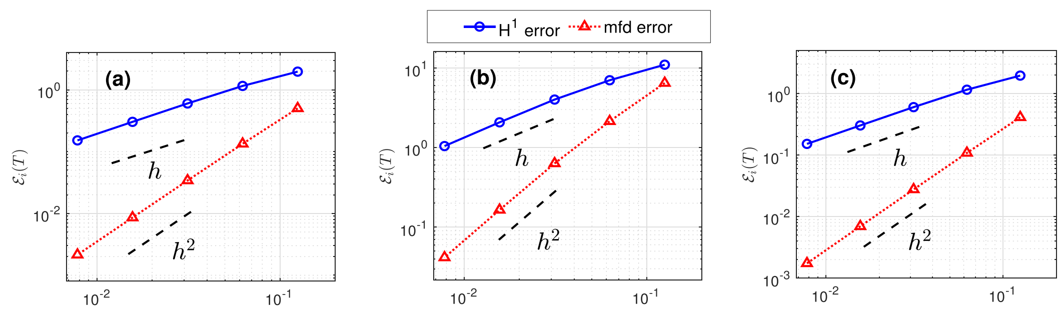

FIG. 3. Numerical errors under different norms of the FEM (5.2) at $T = 1/4$: (a) Case 1; (b) Case 2; (c) Case 3.

where $x^k_{h,\tau}$ represents the solution obtained by the above fully discrete scheme (5.2) or (5.4) with mesh size h and time step τ.

The numerical errors of the FEM (5.2) and the FEM-TM (5.4) are presented in Fig. 3 and Fig. 4, respectively, from which we observe that, for each nonlocal flow with ind or β, the solutions of (5.2) and (5.4) converge linearly in $L^\infty_t H^1_x$, consistent with the theoretical results in Theorems 2 and 3. Moreover, Fig. 4(a) and (b) illustrate that the scheme (5.4) performs equally well for different choices of α. Additionally, we observe that the solution converges quadratically under the manifold distance, which is superior to the theoretical results in Corollary 1 (2) and (3).

5.2 *Evolution of geometric quantities*

In this subsection, we utilize the proposed three methods: FDM (5.1), FEM (5.2) and FEM-TM (5.4) to simulate the nonlocal geometric flows. We are mainly concerned with the evolution of the following geometric quantities: perimeter $L(t)$, relative area loss $\Delta A(t)$ and the mesh ratio function $\Psi(t)$ defined as

$$L(t)|_{t=t_k} = l^k_h, \quad \Delta A(t)|_{t=t_k} = \frac{A^k_h - A^0_h}{A^0_h}, \quad \Psi(t)|_{t=t_k} = \frac{\max_{j=1,\ldots,N} q^k_j}{\min_{j=1,\ldots,N} q^k_j},$$

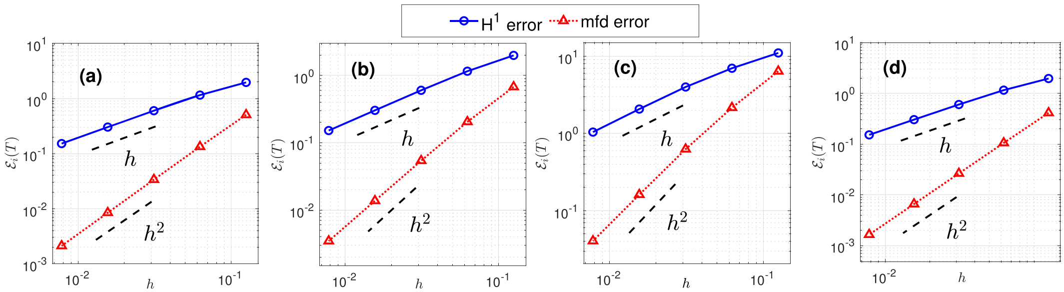

FIG. 4. Numerical errors under different norms of the FEM-TM (5.4) at $T = 1/4$: (a) Case 1 with $\alpha = 1$; (b) Case 1 with $\alpha = 0.5$; (c) Case 2 with $\alpha = 1$; (d) Case 3 with $\alpha = 1$.

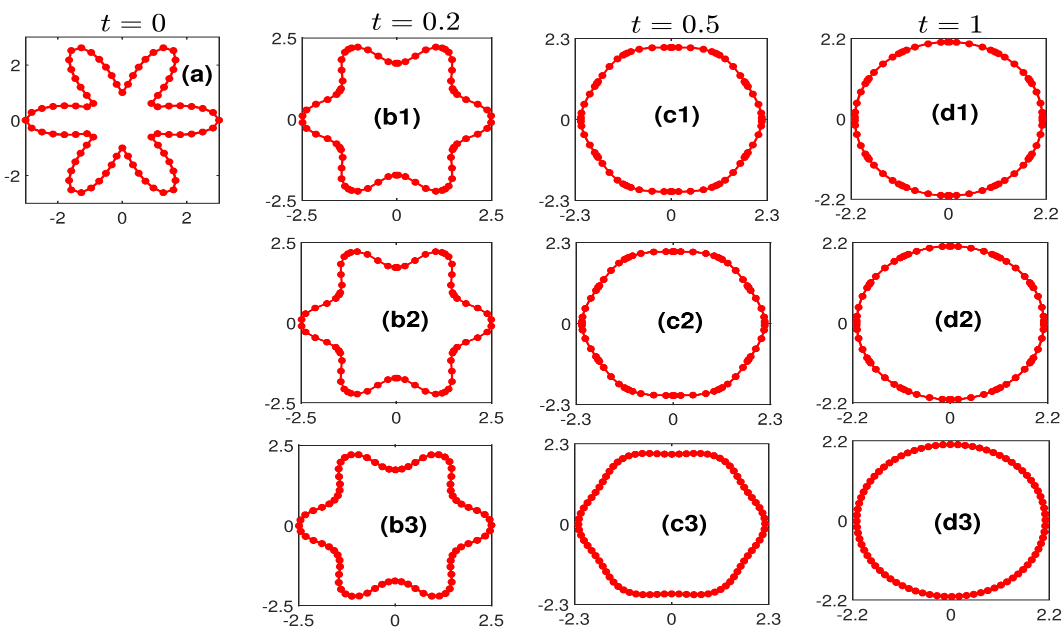

FIG. 5. Snapshots of the curve evolution using the FDM (first row), FEM (second row) and FEM-TM (third row) with $\alpha = 1$ for Case 1. The parameters are chosen as $N = 80$ and $\tau = 1/160$.

where l_h^k and A_h^k are the perimeter and the area of the polygon determined by x_h^k, respectively, and $q_j^k = |x_j^k - x_{j-1}^k|$. Note that for the area of an immersed curve, such as the four-leaf rose, it is treated as a signed area. In morphological evolutions, we primarily focus on the following cases:

Case 1: A flower initial curve parametrized by

$$((2+\cos(6\theta))\cos\theta, (2+\cos(6\theta))\sin\theta)^T, \quad \theta \in [0, 2\pi],$$

with the corresponding flow being the AP-CSF with $f(L) = 2\pi/L$;

Case 2: A four-leaf rose initial curve, with the corresponding flow being the AP-CSF for a nonsimple curve with $f(L) = 2\pi\,\mathrm{ind}/L, \mathrm{ind}(\Gamma) = 3$;

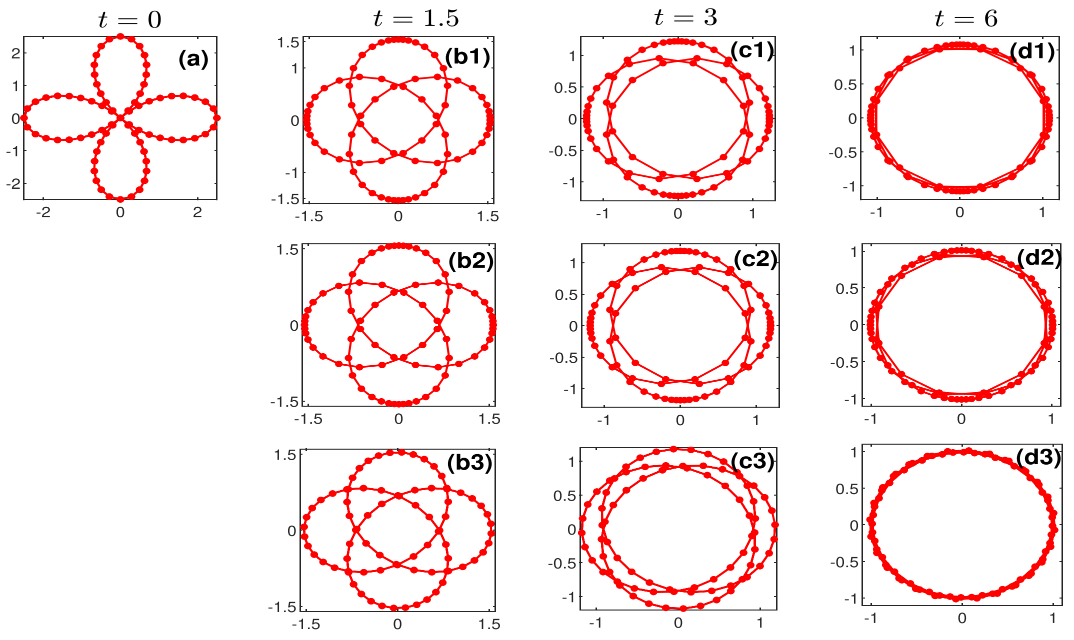

FIG. 6. Snapshots of the curve evolution using the FDM (first row), FEM (second row) and FEM-TM (third row) with $\alpha = 1$ for Case 2. The parameters are chosen as $N = 80$ and $\tau = 1/160$.

Case 3: A 4×1 rectangular initial curve with the corresponding flow being a curve flow with area decreasing rate of π, i.e., $f(L) = (2\pi - \beta)/L$, $\beta = \pi$.

Figures 5–8 depict the comparisons of the three schemes through the evolutions of the solution and geometric quantities for the respective three cases. Based on the observations from Figs 5–8, we can draw the following conclusions:

(i) All of the schemes can evolve the above three cases successfully into their equilibriums, i.e., circle for Case 1, triple circle for Case 2 and a round point for Case 3, which agrees with the theoretical results in Wang & Kong (2014) (cf. Figs 5, 6 and 7).

(ii) For Case 1 and Case 2, the area is conserved numerically up to some precision while the area is decreasing numerically with the rate π for Case 3 (cf. Fig. 8(b)).

(iii) As demonstrated in Fig. 8(c), the FEM-TM redistributes the points during the evolution and ultimately achieves the equidistribution property, i.e., $\Psi(t) \to 1$ as $t \to +\infty$. This coincides the motivation in Section 3.2. In contrast, the FDM and the FEM fail to preserve good mesh quality during the evolution.

We close this section with a numerical example to demonstrate that the parameter α in the FEM-TM (5.4) signifies the velocity of tangential motions. We conduct simulations for Case 1 using the FEM-TM with varying values $\alpha = 0.1, 0.5$ and 1. As depicted in Fig. 9(c), a smaller α leads to a more effective redistribution of the mesh points. Figure 9(b) illustrates that as α approaches 0, the loss of area becomes greater. This indicates that for a fixed set of computational parameters N and τ, a smaller value of α

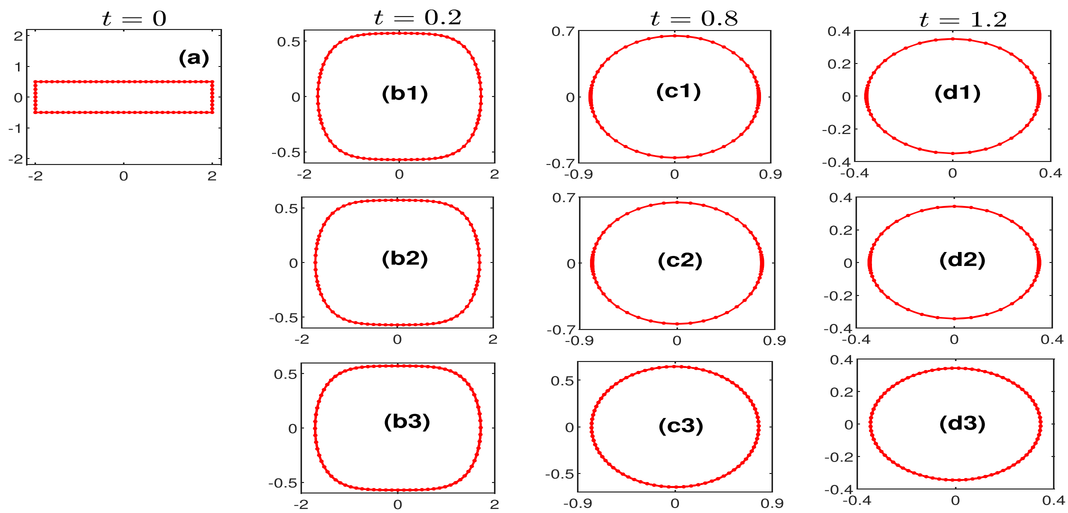

FIG. 7. Snapshots of the curve evolution using the FDM (first row), FEM (second row) and FEM-TM (third row) with $\alpha = 1$ for Case 3. The parameters are chosen as $N = 80$ and $\tau = 1/160$.

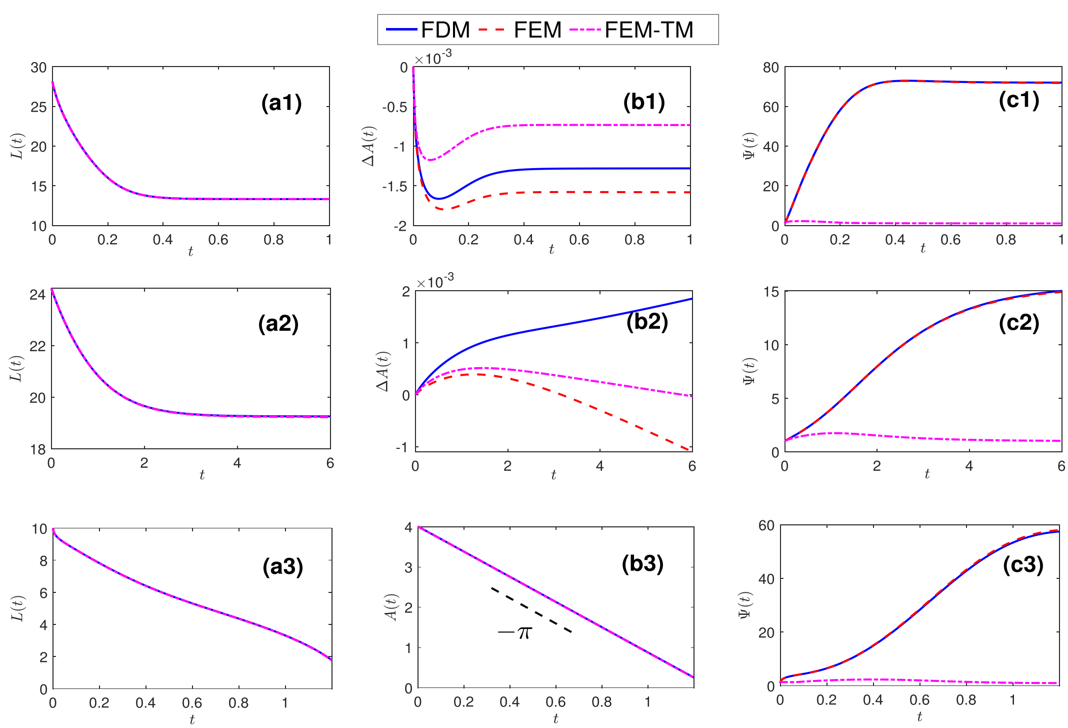

FIG. 8. Evolution of the geometric quantities using the FDM, FEM and FEM-TM with $\alpha = 1$ for Cases 1–3 is illustrated in the first through third rows, respectively. (a) Perimeter $L(t)$; (b) Relative area loss $\Delta A(t)$; (c) Mesh ratio function $\Psi(t)$. The parameters are chosen as $N = 640$ and $\tau = 1/1280$.

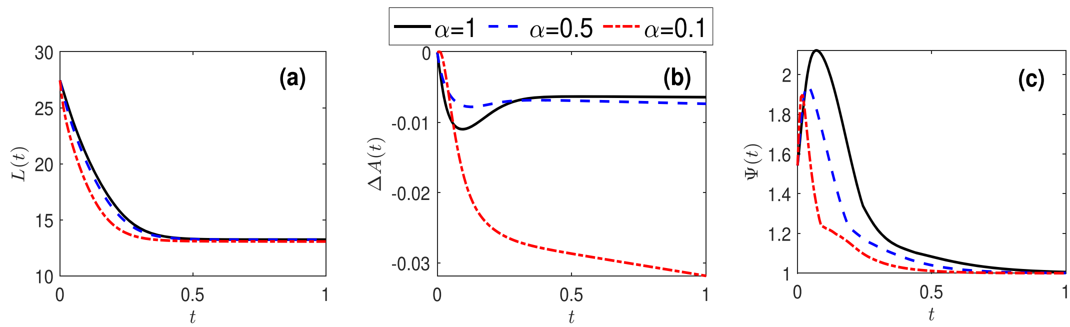

FIG. 9. Evolution of geometric quantities using the FEM-TM with different $\alpha = 0.1, 0.5, 1$ for Case 1. (a) Perimeter $L(t)$; (b) Relative area loss $\Delta A(t)$; (c) Mesh ratio function $\Psi(t)$. The parameters are chosen as $N = 80$ and $\tau = 1/160$.

yields a less accurate simulation, aligning with the findings in Theorem 3, wherein the exponential of $\frac{1}{\alpha}$ is involved in the error estimate.

6. Conclusions

We developed three distinct semidiscrete schemes for simulating some nonlocal geometric flows involving perimeter and the corresponding error estimates were established. Specifically, the FDM exhibits quadratic convergence in H^1, whereas the FEM and the FEM-TM are convergent at the first order in H^1. Furthermore, all three methods demonstrate robust quadratic convergence under manifold distance. Extensive numerical experiments have underscored the superior mesh quality of the FEM-TM compared to FDM and FEM.

It is noteworthy that our proof of the error estimate under manifold distance is not optimal for FEM-TM and FEM. Exploring the possibility of providing a proof of optimal convergence for piecewise linear finite element would be a valuable endeavor.

Funding

W.J. was supported by the National Natural Science Foundation of China Grant (No. 12271414) and the Guangdong Basic and Applied Basic Research Foundation Grant (No. 2024A1515012505), and C.S. and G.Z. were supported by National Key R&D Program of China (Grant No. 2023Y-FA1008902) and the National Natural Science Foundation of China Grant (No. 12201342).

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A. Appendix

In this appendix, we present a convergence result for the fully discrete FEM-TM scheme (5.3). We adapt the approach from Barrett *et al.* (2017) to couple curve evolution with reaction-diffusion. We first need the following assumption for the solution X:

ASSUMPTION A.1. Suppose that the solution of (3.16) with an initial value $X^0 \in H^2(\mathbb{S}^1)$ satisfies $X \in W^{1,\infty}\left([0,T], H^2(\mathbb{S}^1)\right) \cap H^2\left([0,T], H^1(\mathbb{S}^1)\right) \cap L^\infty\left([0,T], H^3(\mathbb{S}^1)\right)$, i.e.,

$$K_3(X) := \|X\|_{W^{1,\infty}([0,T],H^2(\mathbb{S}^1))} + \|X\|_{H^2([0,T],H^1(\mathbb{S}^1))} + \|X\|_{L^\infty([0,T],H^3(\mathbb{S}^1))} < \infty.$$

And there exist constants $0 < C_1 < C_2$ such that (2.2) holds.

THEOREM A.1. Let $X(\xi, t)$ be a solution to (3.16) satisfying Assumption A.1. Assume that the partition of $\mathbb{S}^1$ meets Assumption 3.1. Then, there exists $h_0 > 0$ such that for all $0 < h \le h_0$ and $\tau \le d_0 h$, there exists a unique solution for the FEM-TM scheme (5.3) and it satisfies the following error estimate

$$\max_{k=0,\ldots,m} |X(t_k) - x_h^k|_{H^1}^2 + \sum_{k=1}^{m} \tau \|\partial_t X(t_k) - \delta_\tau x_h^k\|_{L^2}^2 \le Ch^2 + C(1+d_0^2)e^{d_1 C_T} h^2, \tag{A.1}$$

where h_0, C are constants that depend on c_p, c_P, C_1, C_2, α, $K_3(X)$, and f, and $d_1 = C(1+d_0^2)$, $C_T = \int_0^T (1 + \|\partial_{tt} X(\cdot,t)\|_{H^1}^2)\,\mathrm{d}t$.

Proof. Denote

$$e_h^k := X^k - x_h^k = X^k - I_h X^k + I_h X^k - x_h^k = \rho_h^k + \eta_h^k,$$

where $X^k = X(\cdot, t_k)$. We prove the following estimate by induction

$$|\eta_h^k|_{H^1}^2 \le h^2 e^{\theta \int_0^{t_k} \zeta(t)\,\mathrm{d}t}, \qquad h \in (0, h^*], \tag{A.2}$$

where $\zeta(t) = 1 + \|\partial_{tt} X(\cdot,t)\|_{H^1}^2$, θ is a constant independent of h and τ, and will be chosen later, and h^* is chosen small enough such that

$$0 < \frac{C_1}{2} \le |\partial_\xi x_h^{k-1}| \le 2C_2. \tag{A.3}$$

The proof combines the induction (A.2) with an inverse inequality argument similar to Lemma 3 (see also Barrett *et al.* (2017, (3.6), (3.7))). By evaluating (3.17) at time t_k, taking $v = v_h$ in (3.17), and subtracting (5.3) from (3.17), we get

$$\begin{aligned}
&\int_{\mathbb{S}^1} (\alpha \partial_t X(t_k) + (1-\alpha)(\partial_t X(t_k)\cdot \mathcal{N}^k)\mathcal{N}^k)\cdot v_h\, |\partial_\xi X^k|^2 \\
&\quad - I_h\left[(\alpha \delta_\tau x_h^k + (1-\alpha)(\delta_\tau x_h^k \cdot n_h^{k-1}) n_h^{k-1})\cdot v_h\right] |\partial_\xi x_h^{k-1}|^2\,\mathrm{d}\xi + \int_{\mathbb{S}^1}\left(\partial_\xi X^k - \partial_\xi x_h^k\right)\cdot \partial_\xi v_h\,\mathrm{d}\xi \\
&\quad + \int_{\mathbb{S}^1} f(L^k)\mathcal{N}^k \cdot v_h |\partial_\xi X^k|^2 - I_h\left[f(l_h^{k-1}) n_h^{k-1}\cdot v_h\right] |\partial_\xi x_h^{k-1}|^2\,\mathrm{d}\xi = 0,
\end{aligned}$$

where we denote $\mathcal{N}^k = \mathcal{N}(t_k)$. Choosing $v_h = \tau\delta_\tau\eta_h^k$, a straightforward computation yields

$$
\begin{aligned}
&\tau\int_{\mathbb{S}^1} I_h\left[(\alpha\delta_\tau\eta_h^k + (1-\alpha)(\delta_\tau\eta_h^k\cdot n_h^{k-1})n_h^{k-1})\cdot\delta_\tau\eta_h^k\right]|\partial_\xi x_h^{k-1}|^2\,\mathrm{d}\xi + \tau\int_{\mathbb{S}^1}\partial_\xi\eta_h^k\cdot\partial_\xi\delta_\tau\eta_h^k\,\mathrm{d}\xi\\
&= \tau\int_{\mathbb{S}^1} I_h\left[(\alpha\delta_\tau X^k + (1-\alpha)(\delta_\tau X^k\cdot n_h^{k-1})n_h^{k-1})\cdot\delta_\tau\eta_h^k\right]|\partial_\xi x_h^{k-1}|^2\\
&\quad - \tau\left(\alpha\partial_t X(t_k) + (1-\alpha)(\partial_t X(t_k)\cdot\mathcal{N}^k)\mathcal{N}^k\right)\cdot\delta_\tau\eta_h^k|\partial_\xi X^k|^2\,\mathrm{d}\xi\\
&\quad + \tau\int_{\mathbb{S}^1} I_h\left[f(l_h^{k-1})n_h^{k-1}\cdot\delta_\tau\eta_h^k\right]|\partial_\xi x_h^{k-1}|^2 - f(L^k)\mathcal{N}^k\cdot\delta_\tau\eta_h^k|\partial_\xi X^k|^2\,\mathrm{d}\xi - \tau\int_{\mathbb{S}^1}\partial_\xi\rho_h^k\cdot\partial_\xi\delta_\tau\eta_h^k\,\mathrm{d}\xi\\
&=: A_1 + A_2 + A_3.
\end{aligned}
$$

By applying the bound given in (A.3), we can estimate the left-hand side (LHS) as follows:

$$
\mathrm{LHS} \geq \tau\alpha\frac{C_1^2}{4}\|\delta_\tau\eta_h^k\|_{L^2}^2 + \tau(1-\alpha)\frac{C_1^2}{4}\|\delta_\tau\eta_h^k\cdot n_h^{k-1}\|_{L^2}^2 + \frac{1}{2}\left(|\eta_h^k|_{H^1}^2 + |\eta_h^k - \eta_h^{k-1}|_{H^1}^2 - |\eta_h^{k-1}|_{H^1}^2\right). \tag{A.4}
$$

The estimate of A_1 can be found in Barrett *et al.* (2017, (3.13)–(3.16)), which reads as

$$
|A_1| \leq \varepsilon\tau\|\delta_\tau\eta_h^k\|_{L^2}^2 + C(\varepsilon)\left(\tau|\eta_h^{k-1}|_{H^1}^2 + h^2\int_{t_{k-1}}^{t_k}\zeta(t)\,\mathrm{d}t\right) + C(\varepsilon)d_0^2\tau h^2. \tag{A.5}
$$

Moreover, the interpolation estimate and inverse estimate yield the bound of A_3:

$$
|A_3| \leq C\tau\|\partial_\xi\rho_h^k\|_{L^2}\|\partial_\xi\delta_\tau\eta_h^k\|_{L^2} \leq C\tau h^{3/2}\|X^k\|_{H^3}\|\delta_\tau\eta_h^k\|_{L^2} \leq C(\varepsilon)\tau h^2 + \varepsilon\tau\|\delta_\tau\eta_h^k\|_{L^2}^2. \tag{A.6}
$$

The estimate of A_2 involves the nonlocal term and it is important to note that

$$
\begin{aligned}
A_2 &= \tau\int_{\mathbb{S}^1} f(l_h^{k-1})\left(n_h^{k-1} - \mathcal{N}^k\right)\cdot\delta_\tau\eta_h^k|\partial_\xi x_h^{k-1}|^2 + f(l_h^{k-1})\mathcal{N}^k\cdot\delta_\tau\eta_h^k\left(|\partial_\xi x_h^{k-1}|^2 - |\partial_\xi X^k|^2\right)\,\mathrm{d}\xi\\
&\quad + \tau\int_{\mathbb{S}^1}\left(f(l_h^{k-1}) - f(L^k)\right)\mathcal{N}^k\cdot\delta_\tau\eta_h^k|\partial_\xi X^k|^2\,\mathrm{d}\xi\\
&=: A_{21} + A_{22}.
\end{aligned}
$$

By recalling (A.3), using Barrett *et al.* (2017, (3.18), (3.20)), along with the Lipschitz property of f and Young's inequality, we are led to

$$
|A_{21}| \leq C\tau\left(h + |\eta_h^{k-1}|_{H^1}\right)\|\delta_\tau\eta_h^k\|_{L^2} \leq C(\varepsilon)\tau h^2 + C(\varepsilon)\tau|\eta_h^{k-1}|_{H^1}^2 + \varepsilon\tau\|\delta_\tau\eta_h^k\|_{L^2}^2, \tag{A.7}
$$

$$
\begin{aligned}
|A_{22}| &\leq C\tau|f(l_h^{k-1}) - f(L^{k-1})|\|\delta_\tau\eta_h^k\|_{L^2} + C\tau|f(L^k) - f(L^{k-1})|\|\delta_\tau\eta_h^k\|_{L^2}\\
&\leq C\tau\|\partial_\xi X^{k-1} - \partial_\xi x_h^{k-1}\|_{L^2}\|\delta_\tau\eta_h^k\|_{L^2} + C\tau^2\|\delta_\tau\eta_h^k\|_{L^2}\\
&\leq C\tau\left(|\eta_h^{k-1}|_{H^1} + d_0 h\right)\|\delta_\tau\eta_h^k\|_{L^2}\\
&\leq C(\varepsilon)d_0^2\tau h^2 + C(\varepsilon)\tau|\eta_h^{k-1}|_{H^1}^2 + \varepsilon\tau\|\delta_\tau\eta_h^k\|_{L^2}^2,
\end{aligned} \tag{A.8}
$$

where we used the assumption that $\tau \leq d_0 h$. Finally, by combining the above estimates (A.4)–(A.8), choosing ε sufficiently small and $\theta = C(1+d_0^2) =: d_1$, and applying the induction hypothesis together with the fact that $\zeta \geq 1$, we get

$$\begin{aligned}
&\tau\frac{C_1^2}{4}\left\|\delta_\tau \eta_h^k\right\|_{L^2}^2 + \tau\frac{C_1^2}{4}\|\delta_\tau\eta_h^k\cdot n_h^{k-1}\|_{L^2}^2 + |\eta_h^k|_{H^1}^2 + |\eta_h^k-\eta_h^{k-1}|_{H^1}^2 \\
&\leq |\eta_h^{k-1}|_{H^1}^2 + C\tau|\eta_h^{k-1}|_{H^1}^2 + Cd_0^2h^2\int_{t_{k-1}}^{t_k}\zeta(t)\,\mathrm{d}t \\
&\leq h^2 e^{\theta\int_0^{t_{k-1}}\zeta(t)\,\mathrm{d}t}\left(1 + C(1+d_0^2)\int_{t_{k-1}}^{t_k}\zeta(t)\,\mathrm{d}t\right) \\
&= h^2 e^{\theta\int_0^{t_{k-1}}\zeta(t)\,\mathrm{d}t}\left(1+\theta\int_{t_{k-1}}^{t_k}\zeta(t)\,\mathrm{d}t\right) \\
&\leq h^2 e^{\theta\int_0^{t_{k-1}}\zeta(t)\,\mathrm{d}t} e^{\theta\int_{t_{k-1}}^{t_k}\zeta(t)\,\mathrm{d}t} = h^2 e^{\theta\int_0^{t_k}\zeta(t)\,\mathrm{d}t}. \qquad \text{(A.9)}
\end{aligned}$$

Thus, we conclude the induction proof. Therefore, by applying the standard interpolation estimate and (A.2), we get

$$\max_{k=0,\ldots,m}|X^k - x_h^k|_{H^1}^2 \leq (C + e^{d_1 C_T})h^2, \qquad C_T := \int_0^T \zeta(t)\,\mathrm{d}t. \qquad \text{(A.10)}$$

Summing (A.9) from $k = 1,\ldots,m$ and noting the interpolation estimate, we derive

$$\sum_{k=1}^m \tau\|\partial_t X(t_k) - \delta_\tau x_h^k\|_{L^2}^2 \leq Ch^2 + C\sum_{k=1}^m \tau h^2 e^{\theta\int_0^{t_k}\zeta(t)\,\mathrm{d}t} + Cd_0^2h^2e^{d_1C_T} \leq C(1+d_0^2)e^{d_1C_T}h^2. \quad \text{(A.11)}$$

The proof (A.1) is completed by combining (A.10) and (A.11). □